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ALGEBRAIC STRUCTURES AND PROBABILITY

ELLIOTT
FRYER
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ERRATA

CORRECTIONS TO THE TEXT OF *ALGEBRAIC STRUCTURES AND PROBABILITY*

Chapter 7

EXERCISE 7.4 (page 165)

Question (2) should read:

Find $z + \bar{z}$ and $z - \bar{z}$ for each complex number in question (1).

Question (10) should read:

Prove that $\overline{\overline{z_1}} = z_1$ for any complex number z_1.

CORRECTIONS TO THE ANSWERS

Chapter 1

EXERCISE 1.9 (page 25)

6. (i) 512. (ii) 336. **8.** 216 and 6.

Chapter 2

EXERCISE 2.6 (page 50)

5. 64,800.

REVIEW EXERCISE 2 (page 52)

7. (b) 20,160. **26.** (d) $\{12, 3\}$. **28.** 163.

Chapter 3

EXERCISE 3.2 (page 60)

2. (a) $6 \sum_{k=1}^{n} k + 4n$.

EXERCISE 3.5 (page 74)

5. $\frac{n}{2}(n+1)(n^2+n-1)$.

EXERCISE 3.6 (page 77)

5. $\frac{3}{3-2x} + \frac{1}{1+x}$. **16.** $\frac{-3n(3n+5)}{4(3n-1)(3n+2)}$.

REVIEW EXERCISE 3 (page 78)

3. (a) $\sum_{i=1}^{6} (2i+3)$.

5. (a) $\frac{n}{2}(3n+1)$. (b) $\frac{n}{2}(n+3)$. (e) $\frac{n}{4}(n-1)(n+2)(n-3)$.

10. (b) $\frac{n}{2(3n+2)}$. (c) $\frac{5n}{4(5n+4)}$.

Chapter 4

EXERCISE 4.3 (page 90)

1. (i) $\binom{n}{r} 2^r a^{3n-5r}$.

6. -84; -126; no term of form ka^{-10}.

REVIEW EXERCISE 4 (page 91)

9. 32,805; 252.

Chapter 5

EXERCISE 5.3 (page 105)

1. (j) $\frac{7}{18}$.

6. (c) $\frac{9}{24} = \frac{3}{8}$.

EXERCISE 5.8 (page 125)

1. (c) $\frac{671}{1296}$.

REVIEW EXERCISE 5 (page 128)

5. (b) $\frac{21}{46}$.

13. (d) $\frac{73}{729}$.

14. $\frac{821}{3125}$.

Chapter 6

EXERCISE 6.3 (page 141)

6. (a) $(\frac{3}{2}, 60°)$, $(\frac{3}{2}, 300°)$, $(0, 0°)$.

Chapter 7

EXERCISE 7.3 (page 160)

16. (a) $1 + 0i$.

EXERCISE 7.5 (page 167)

3. (h) $\dfrac{\sqrt{3}}{5} - \dfrac{i\sqrt{2}}{5}$.

EXERCISE 7.6 (page 170)

2. (a) $\dfrac{1 \pm \sqrt{47}i}{12}$.

EXERCISE 7.7 (page 175)

4. (f) $-1.6904 + 3.6252i$.

EXERCISE 7.10 (page 183)

1. (f) $0 + 2i$, $\sqrt{3} - i$, $-\sqrt{3} - i$.

(o) $\pm\left(\dfrac{\sqrt{6}}{2} - \dfrac{i\sqrt{2}}{2}\right)$.

Chapter 8

EXERCISE 8.13 (page 226)

3. (d) $A^{-1} - \begin{bmatrix} \frac{2}{3} & -1 \\ \frac{1}{3} & 0 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} \frac{2}{27} & \frac{5}{27} \\ \frac{5}{27} & \frac{-1}{27} \end{bmatrix}$.

EXERCISE 8.15 (page 233)

1. The following systems of Exercise 8.14 are fields. 1(c), (g), (i), (k), (m): (5).

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ALGEBRAIC
STRUCTURES
AND
PROBABILITY

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PREFACE

In this text the authors have attempted to introduce a judicious blending of the set theoretical approach and the more traditional approach to the various topics. The set theoretical approach can be very useful in demonstrating relationships between apparently unrelated topics and, from this point of view, is a powerful mathematical tool. However, overindulgence in set theory simply for the sake of using sets may often cause basically simple ideas to appear much more complicated than they actually are.

Study of Chapters 6, 7, and 8, may usefully be delayed until the authors companion volume entitled *Vectors and Matrices* has been studied. This is not essential since these chapters are complete in themselves, but a familiarity with *Vectors and Matrices* may enable the student to study these chapters more quickly.

Definitions and key points in the various chapters have been printed in red. In addition some problems in certain exercises have been numbered in red. These tend to be more difficult than the other problems and might be omitted on a first reading.

The authors wish to express their thanks to Mr. J. W. Cisarchuk for his assistance in the preparation of the answers to the exercises.

H. A. E.
K. D. F.
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Chapter 1

THE ALGEBRA OF SETS

1.1. Sets and Subsets

The term set is regarded as one of the basic undefined ideas of mathematics. In geometry, the terms point, line, and plane were undefined and their properties were established by means of the axioms. We did find it useful, however, to have some intuitive idea of what we understood by a point, a line, or a plane. In the same way, we find it useful to establish an intuitive idea of what we mean by a set by using what we might call a "pseudo-definition".

A *set* is a collection of objects from a specified universe.

A set is *well-defined* if each member of the universe is either in the set or not in the set.

We should note that our "definition" lacks one of the attributes of a proper definition since the words set and collection are, for all practical purposes, synonymous. We are really using the word set to define itself.

It is important in set theory to have the universe or universal set (U) of any discussion clearly determined. If we are asked for $\{x \mid 1 < x < 2\}$, we cannot give any sensible answer since the universe (in this case, the replacement set for the variable x) has not been specified. If $U = I$, there is no member of the set of integers which satisfies the inequalities and so

$$\{x \in I \mid 1 < x < 2\} = \phi \quad \text{(the empty set or null set).}$$

If $U = Ra$, there is an infinite number of rational numbers which satisfy the inequalities and our solution set is the set of all rationals between 1 and 2.

In many instances, the universe may be obvious from the context and so not be specifically stated. In $\{x \mid x + 1 = 5\}$, if we understand that the universe is a particular set of numbers, we may omit any specific statement to that effect.

So frequently are certain sets of numbers used that special symbols are used to designate these sets. We first give the symbols used in this text, followed by a description of the symbol, and finally symbols that may be used in some other texts.

$$\begin{aligned}
N &= \{\text{all natural numbers}\} = \{1, 2, 3, \cdots\} \\
W &= \{\text{all whole numbers}\} = \{0, 1, 2, 3, \cdots\} = N_0 \\
I &= \{\text{all integers}\} = \{\cdots, -2, -1, 0, +1, +2, \cdots\} = Z \\
I^+ &= \{\text{all positive integers}\} = Z^+ \\
I^- &= \{\text{all negative integers}\} = Z^- \\
Ra &= \{\text{all rationals}\} = Q \\
Re &= \{\text{all real numbers}\} = R = \Re \\
C &= \{\text{all complex numbers}\} = \mathfrak{C}
\end{aligned}$$

If a set A has the property that every element of A is also an element of B, then A is a *subset* of B. This is written as

$$A \subseteq B \quad (A \text{ is contained in or is equal to } B)$$

or

$$B \supseteq A \quad (B \text{ contains } A \text{ or is equal to } A).$$

If every element of A is an element of B but some element of B is not an element of A, then A is a *proper subset* of B. This is written as $A \subset B$ or $B \supset A$. We note that any set is a subset of the universe $(A \subseteq U)$. If A is not identical with U, then $A \subset U$.

We also consider the empty set ϕ to be a subset of every set.

If x is an element of (belongs to) a set A, we write $x \in A$. If x is not an element of A, we write $x \notin A$.

While our work in this chapter will deal mainly with finite sets, there are some properties of infinite sets which may be of interest. The natural numbers can be put into a one-to-one correspondence with a subset of the integers, the positive integers,

$$1 \rightarrow +1, \quad 2 \rightarrow +2, \quad \cdots, \quad n \rightarrow +n, \quad \cdots.$$

Because of the similar properties of N and I^+, we could say that $N \subset I$.

It is also possible, however, to put the set of all integers into a 1-1 correspondence with the set of natural numbers.

$$\begin{array}{cccccccccccccc}
\cdots, & -n & ,\cdots, & -3, & -2, & -1, & 0, & +1, & +2, & +3, & \cdots, & +n, & \cdots \\
\cdots, & \downarrow & \cdots, & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots, & \downarrow & \cdots \\
\cdots, & 2n+1, & \cdots, & 7, & 5, & 3, & 1, & 2, & 4, & 6, & \cdots, & 2n, & \cdots
\end{array}$$

We may, therefore, define an *infinite set* as a set that can be put into a 1-1 correspondence with a proper subset of itself. In this way, the set I is an infinite set and, in a similar manner, we can show that $N, W, Ra, Re, C,$ are infinite sets.

EXERCISE 1.1

1. If $U = I$, state which of the following are sets in this universe.
   (a) {all prime integers}
   (b) $\{+1, +3, +5, +7, +9\}$
   (c) $\{(1,2), (1,3), (2,5), (3,-7), (-3,-5)\}$
   (d) $\{x \in I \mid 2x - 1 > 3\}$
   (e) $\{y \mid y = \frac{x}{2} + 3,\ x \text{ is an even integer}\}$
   (f) $\{y \mid y = \frac{x}{2} + 3,\ x \in I\}$
2. Let U be the set of positive integers less than 5.
   (a) Form a new universe V of ordered pairs (x, y), where $x \in U$ and $y \in U$. How many elements are in V?
   (b) W is the set of ordered pairs (x, y) such that $x \in U$, $y \in U$, and $x \neq y$. How many elements are in W?
   (c) Are the following statements true?
   $$V \subset U. \qquad W \subset U. \qquad W \subset V.$$
3. $A = \{1, 2\}$, $B = \{\{1, 2, 3\}, \{4, 5\}, 6\}$, $C = \{\{2\}, \{1, 2\}\}$
   State whether the following are true or false.
   (a) $2 \in A$, $2 \in B$, $2 \in C$ (b) $A \in B$, $A \in C$
4. Form sets A, B, C, such that $A \in B$, $B \in C$, and $A \notin C$.
5. Prove that N and Ra are infinite sets.

1.2. Operations on Sets

In this section we will assume that all sets in a particular discussion are sets from the same universe and, unless it is necessary, we will not specify the universe.

The *union* of the sets A and B, symbolized by $A \cup B$ (read "A union B" or "A cup B"), is the set of all elements that are members of either A or B.

Note that here the word "or" is used in the inclusive sense; that is, in A or B means in either A or B or both.

Example 1. If $A = \{a, b, c, d\}$, $B = \{b, c, d, e, f\}$, then

$$A \cup B = \{a, b, c, d, e, f\}.$$

The *intersection* of the sets A and B, symbolized by $A \cap B$ (read "A intersection B" or "A cap B"), is the set of all elements that are members of both A and B. In Example 1, $A \cap B = \{b, c, d\}$.

Example 2. If $X = \{1, 3, 5\}$, $Y = \{\text{all even integers}\}$, then

$$X \cap Y = \phi .$$

In this case the sets X and Y are said to be *disjoint.*

The operations of union and intersection are *binary operations* on sets since they form a new set from two given sets belonging to the universe. In this respect, and in other ways, they may be regarded as analagous to the arithmetical operations of addition and multiplication.

On the other hand the operation of complementation, which we consider next, is a *unary operation* since it forms a new set from *one* given set belonging to the universe.

The *complement* of a set T is that set of elements in U which are not members of T. The complement of T is written as T' (read T prime).

Example 3. If $U = \{1, 2, 3, \cdots, 10\}$ and $A = \{2, 4, 6, 8, 10\}$, then

$$A' = \{1, 3, 5, 7, 9\} .$$

Example 4. If

$$\begin{aligned} U &= \{1, 2, 3, \cdots, 20\} \text{ and} \\ A &= \{x \in U \mid x \text{ is a prime number}\}, \\ B &= \{2, 5, 8, 11, 14, 17, 20\}, \\ C &= \{x \in U \mid x \text{ is even}\}, \end{aligned}$$

state the elements of the sets

$$A \cap B, \quad A \cup C, \quad A', \quad B', \quad C', \quad (A \cup C)', \quad A' \cap C'.$$

Solution:

$$\begin{aligned} A \cap B &= \{2, 5, 11, 17\} \\ A \cup C &= \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20\} \\ A' &= \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\} \\ B' &= \{1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19\} \\ C' &= \{1, 3, 5, \cdots, 19\} \\ (A \cup C)' &= \{1, 9, 15\} \\ A' \cap C' &= \{1, 9, 15\} \end{aligned}$$

EXERCISE 1.2

1. If $A = \{a, b, c, d\}$, $B = \{c, d, e\}$, $C = \{e, f, g, h\}$, state the elements of the sets $A \cup C, \quad B \cap A, \quad B \cap C, \quad B \cap (A \cup C), \quad (B \cap A) \cup (B \cap C)$.

2. If $U = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{10}\}$, $A = \{\frac{1}{2}, \frac{1}{5}, \frac{1}{8}\}$, $B = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$, $C = \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}$, state the elements of the sets
   $A \cap C$, $C \cap A$, $A \cup B$, $B \cup A$, A', B', C', $(A \cup B)'$, $A' \cap B'$, $A' \cap B$, $A' \cap (A \cup B)$.

3. If $U = \{x \in I \mid -5 < x < 5\}$ and $P = \{x \in I \mid -2 < x < 3\}$, state the elements of the sets
   $$P', \quad P' \cap P, \quad P' \cup P, \quad P \cap U, \quad P \cup U.$$

4. If $X = \{-3, -2, -1, 0, 1, 2, 3\}$, $Y = \{-5, -4, -3, -2, -1\}$, $Z = \{-2, -1, 0, 1, 2\}$, show that
   (a) $X \cap (Y \cap Z) = (X \cap Y) \cap Z$,
   (b) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$,
   (c) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,
   (d) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

5. If $U = \{1, 2, 3, \cdots, 10\}$, $A = \{x \in U \mid x \text{ is a prime}\}$, $B = \{x \in U \mid x \text{ is odd}\}$, show that
   (a) $A' \cap B' = (A \cup B)'$,
   (b) $A' \cup B' = (A \cap B)'$.

6. If $U = I^+$, $P = \{x \in I^+ \mid x \text{ is a prime}\}$, $Q = \{x \in I^+ \mid x \text{ is even}\}$, describe the sets or list the elements of the sets
   $$P', \quad Q', \quad P \cap Q, \quad P \cup Q.$$

7. If $A = \{(x, y) \mid x + 2y = 3,\ x,\ y \in Re\}$, $B = \{(x, y) \mid x + y = 1,\ x,\ y \in Re\}$, state the elements of $A \cap B$.

8. If $X = \{(x, y) \mid x + 2y = 3,\ x,\ y, \in Re\}$, $Y = \{(x, y) \mid 3x + 6y = 7,\ x,\ y \in Re\}$, state the elements of $X \cap Y$.

1.3. Laws of Intersection and Union

A useful device to assist our consideration of relationships that may exist between the subsets of a given universe is a Venn diagram. The universal set is represented by the set of points inside and on the boundary of a simple closed curve (usually a rectangle) and a set A in U by the points inside and on the boundary of a second simple closed curve (usually a circle) inside the rectangle. In Figure 1.1, U and A are both labelled but, in future, we will assume that the set of points inside and on the rectangle represent the universe under discussion and will omit the label U.

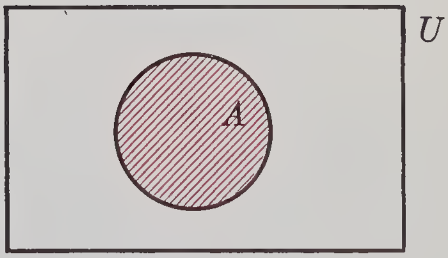


Figure 1.1

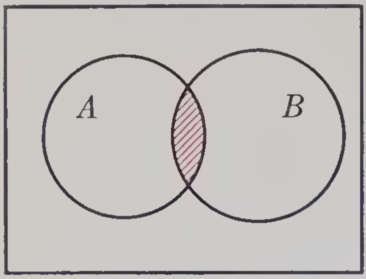


Figure 1.2

$A \cap B$

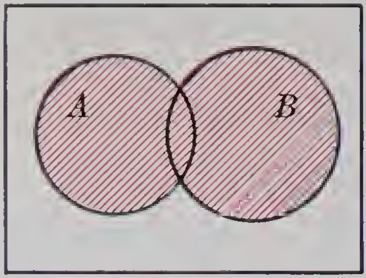


Figure 1.3

$A \cup B$

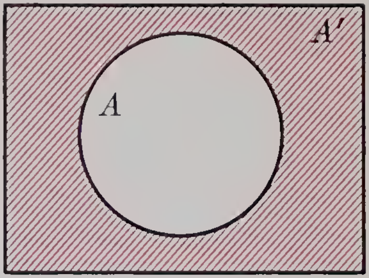


Figure 1.4

A'

The intersection $A \cap B$ of two sets A and B is represented by the common region enclosed by the two intersecting circles as in the shaded part of Figure 1.2.

The union $A \cup B$ of two sets is represented by the sets of points in A or B or both as in the shaded part of Figure 1.3.

The complement A' of a set is represented by the set of all points in U but not in A as in the shaded part of Figure 1.4. (Note that the circle is *not* part of A'.)

If $A \cap B = \phi$, then A and B are *disjoint* and in the Venn diagram the circles enclosing the sets A and B do not intersect as in Figure 1.5.

If $A \subseteq B$, the circle enclosing the set A will be completely contained in the circle enclosing the set B as in Figure 1.6.

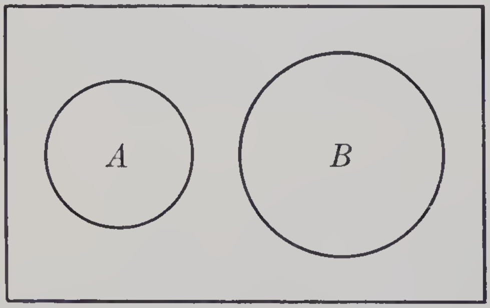


Figure 1.5

$A \cap B = \phi$

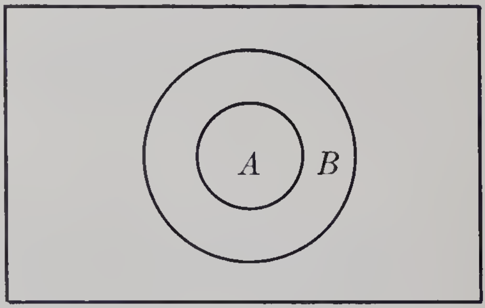


Figure 1.6

$A \subset B$

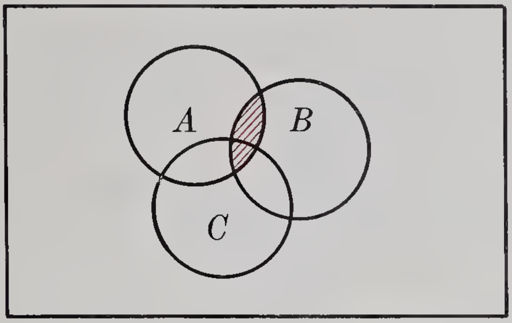
Using these six Venn diagrams as a basis, we can produce a geometrical interpretation for any expression involving sets and set operations. It should be noted, however, that Venn diagrams *do not prove* the truth of a relationship between sets; they only illustrate its plausibility. Also, the use of Venn diagrams to illustrate relationships between more than three sets becomes very complicated and it is difficult to see clearly the relationship being studied.

From Figures 1.2 and 1.3 it is immediately obvious that

$$\text{(i)} \quad A \cap B = B \cap A \qquad \text{and} \qquad \text{(ii)} \quad A \cup B = B \cup A\,.$$

Thus each of the operations of set intersection and set union is a *commutative* operation. The analogy between the set operations $\cap$ and $\cup$ and the arithmetical operations × and + is apparent. Arithmetically, if a and b are numbers, then

$$a \times b = b \times a \qquad \text{and} \qquad a + b = b + a\,.$$



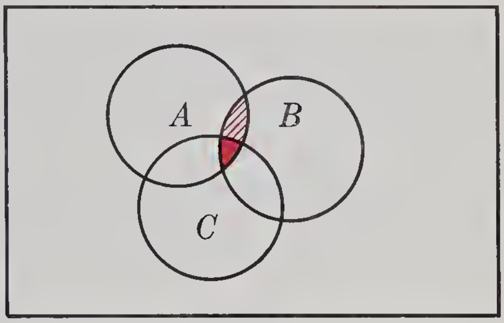
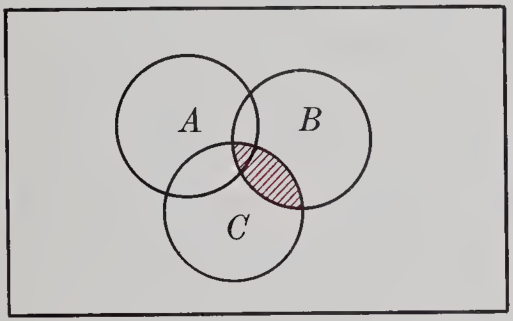


Figure 1.7 (i)

$A \cap B$

Figure 1.7 (ii)

$(A \cap B) \cap C$



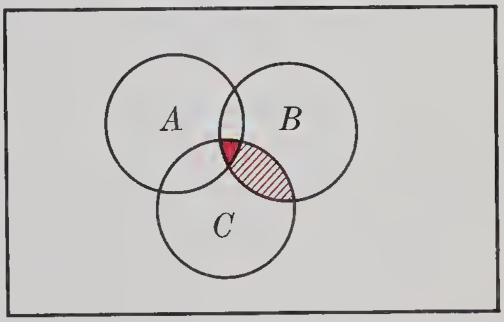


Figure 1.7 (iii)

$B \cap C$

Figure 1.7 (iv)

$A \cap (B \cap C)$

The Venn diagrams in Figure 1.7 illustrate the result of applying the operation of set intersection to three sets A, B, and C.

In Figure 1.7(*i*), the shaded region represents the set $A \cap B$. The doubly-shaded region in Figure 1.7(*ii*) represents the result of the operation of intersection on the new set $A \cap B$ and the set C, that is, $(A \cap B) \cap C$.

In Figures 1.7(*iii*) and 1.7(*iv*), the set $B \cap C$ is formed first and again the doubly-shaded region in Figure 1.7(*iv*) represents the set $A \cap (B \cap C)$.

Since the doubly-shaded regions in Figures 1.7(*ii*) and 1.7(*iv*) are the same, we may conclude that

$$(A \cap B) \cap C = A \cap (B \cap C)\,.$$

This result was also shown in a special case in question (4a) of Exercise 1.2. It would appear to be true in all cases that the operation of set intersection is *associative*. Again we may compare the associative law of set intersection with the associative law of multiplication in arithmetic, where

$$(a \times b) \times c = a \times (b \times c)\,.$$

Generally in arithmetic the use of the associative law permits the omission of parentheses and we simply write $a \times b \times c$. Similarly, for set intersection, we may write $A \cap B \cap C$.

In the same way, we may demonstrate with Venn diagrams in Figures 1.8(*i*), (*ii*), (*iii*), (*iv*) that the operation of union is associative.

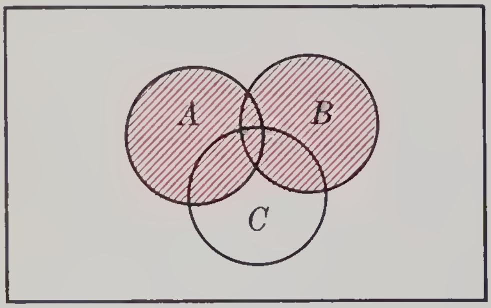


Figure 1.8(i)

$A \cup B$

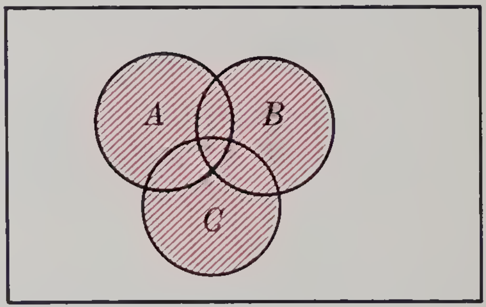


Figure 1.8(ii)

$(A \cup B) \cup C$

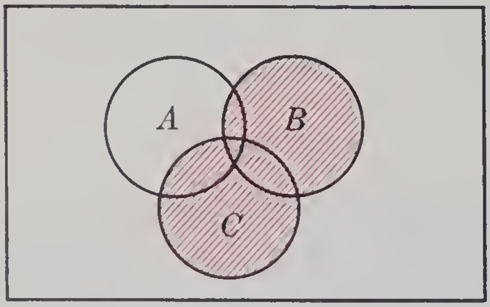


Figure 1.8(iii)

$B \cup C$

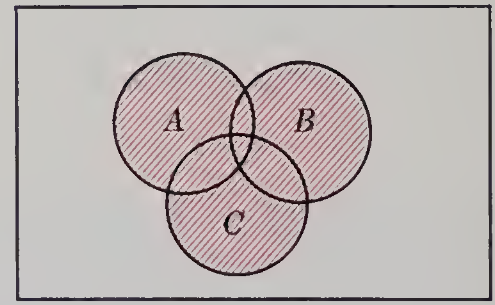


Figure 1.8(iv)

$A \cup (B \cup C)$

$$(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$$

which we may compare with

$$(a + b) + c = a + (b + c) = a + b + c$$

for the addition of numbers.

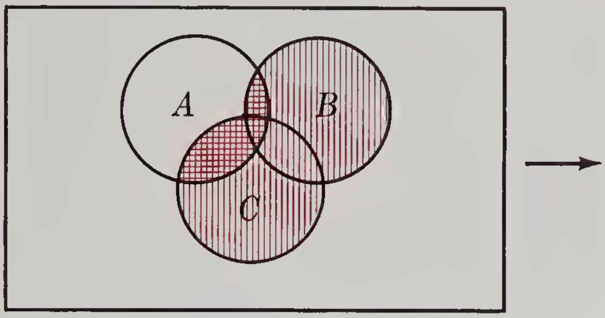


Figure 1.9(i)

$A \cap (B \cup C)$

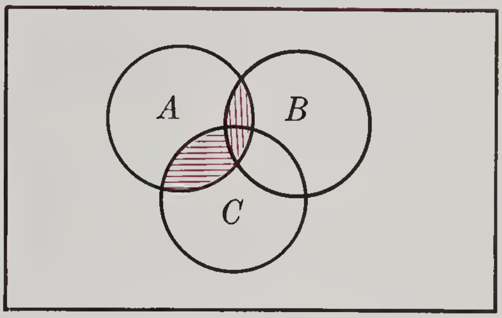


Figure 1.9(ii)

$(A \cap B) \cup (A \cap C)$

In Figure 1.9(*i*), the vertically-shaded region represents the set $B \cup C$; the horizontally-shaded region represents the intersection of A with $B \cup C$, or $A \cap (B \cup C)$.

In Figure 1.9(*ii*), the vertically-shaded region represents the set $A \cap B$ and the horizontally-shaded region, the set $A \cap C$. The union of these two sets is the set $(A \cap B) \cup (A \cap C)$ and we see that this is the same region as $A \cap (B \cup C)$. The Venn diagrams illustrate the *distributive law* of intersection over union.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) .$$

We may compare this with the arithmetical distributive law of multiplication over addition.

$$a(b + c) = ab + ac .$$

In Exercise 1.3, question (1) asks for the illustration, by means of Venn diagrams, of the distributive law of union over intersection.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) .$$

Here we see that the arithmetical analogy is false. In arithmetic, addition is not distributive over multiplication.

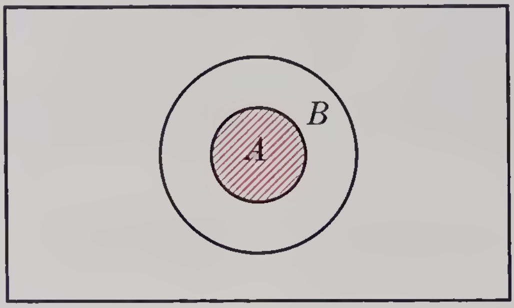
$$a + (b \times c) \neq (a + b) \times (a + c) .$$

We see, therefore, that the arithmetical analogy, while it is useful in certain cases, should be used with great caution. The two systems are not *isomorphic*. That is, briefly and rather loosely, they do not have exactly the same structure.

Example 1. Illustrate, by means of a Venn diagram, that

$$\text{if } A \subseteq B, \text{ then } A \cap B = A.$$

Solution:

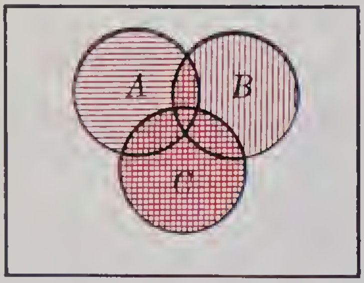


$A \cap B$ is the shaded region. $A \cap B = A$ if $A \subseteq B$.

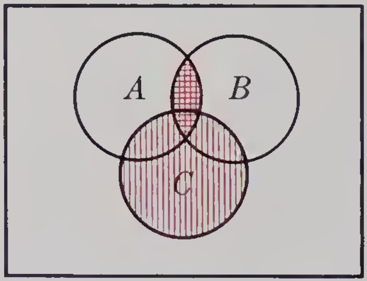
Example 2. Illustrate, by means of Venn diagrams, that

$$(A \cap B) \cup [(A \cup C) \cap (B \cup C)] = (A \cap B) \cup C .$$

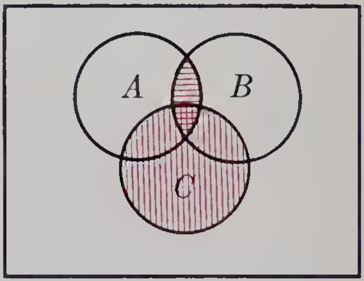
Solution:



$(A \cup C) \cap (B \cup C)$
is
the doubly-shaded region.



$(A \cap B) \cup [(A \cup C) \cap (B \cup C)]$
is
all the shaded region.



$(A \cap B) \cup C$
is
all the shaded region.

Therefore

$$(A \cap B) \cup [(A \cup C) \cap (B \cup C)] = (A \cap B) \cup C\,.$$

EXERCISE 1.3

Illustrate the following identities by means of Venn diagrams.

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. If $A \subseteq B$, then $A \cup B = B$.
3. $A \cap U = A$.
4. $A \cup U = U$.
5. $A \cap (A \cup B) = A$.
6. $A \cup (A \cap B) = A$.
7. $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.
8. $(A \cup B) \cap (A \cap B) = A \cap B$.
9. $(A \cup B) \cup (A \cap B) = A \cup B$.
10. $(A \cap B) \cup (A \cap C) \cup (B \cap C) = (A \cup B) \cap (B \cup C) \cap (A \cup C)$.

1.4. Laws of Complementation

In this section we will make use of Venn diagrams to illustrate various laws of complementation and combinations of these laws with the laws of intersection and union.

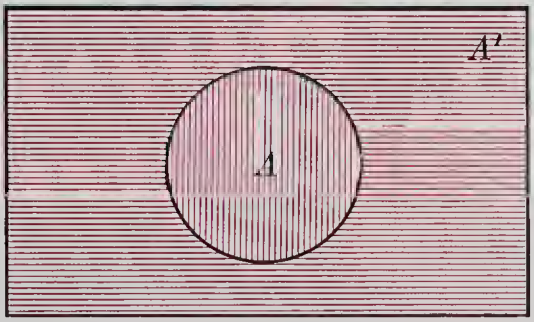


Figure 1.10

$$A \cap A' = \phi \quad \text{and} \quad A \cup A' = U$$

In Figure 1.10, the region which represents the set A is shaded vertically and the region which represents A', the complement of A, is shaded horizontally.

We note from the definitions in Section 1.1 that the points on the circle are in A and not in A'. We see that there are no elements common to both A and A'. This, of course, is obvious from the definition of A'. Hence

$$A \cap A' = \phi .$$

Also, from Figure 1.10, we see that the union of A and A' includes all elements of the universe and hence

$$A \cup A' = U .$$

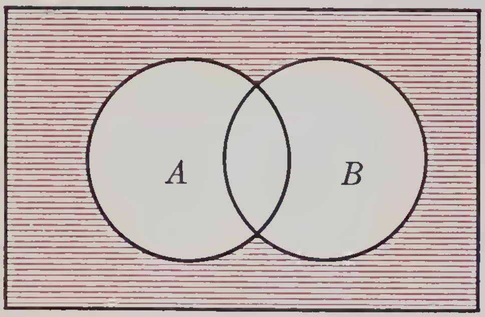


Figure 1.11(i)

$(A \cup B)'$

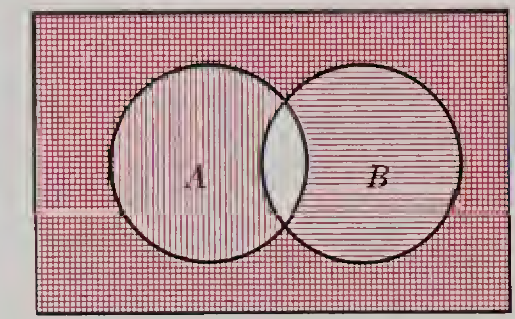


Figure 1.11(ii)

$A' \cap B'$

In Figure 1.11(*i*), the unshaded region represents the set $A \cup B$, and therefore the shaded region will represent $(A \cup B)'$.

In Figure 1.11(*ii*), the vertically-shaded region represents the set B', and the horizontally-shaded region represents the set A'. The doubly-shaded region, therefore, represents the set $A' \cap B'$. We note that the doubly-shaded region in Figure 1.11(*ii*) is the same region as the shaded region in Figure 1.11(*i*). The Venn diagrams illustrate the law that the complement of a union is the intersection of the complements.

$$(A \cup B)' = A' \cap B'.$$

In Exercise 1.4, question (1), we are asked to use Venn diagrams to illustrate the law that the complement of an intersection is the union of the complements.

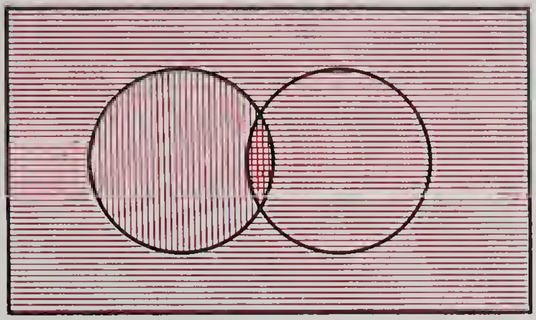
$$(A \cap B)' = A' \cup B' .$$

These two laws are known as the De Morgan Laws.

Example 1. Use a Venn diagram to illustrate that

$$A \cap (A' \cup B) = A \cap B .$$

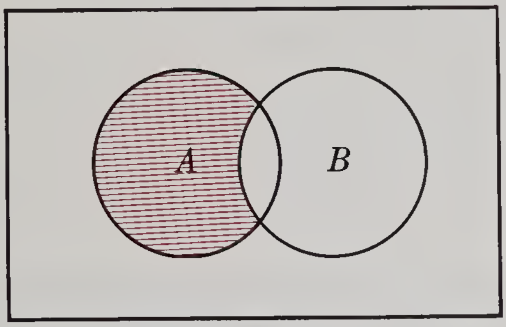
Solution:



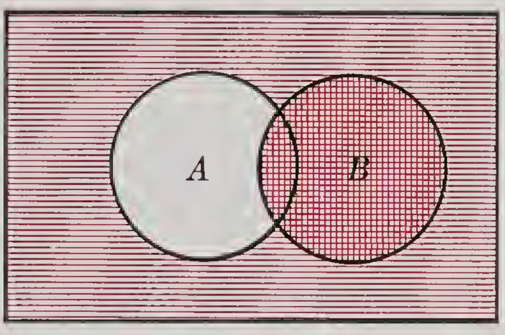
▤ $(A' \cup B)$ ▥ A ▦ $A \cap (A' \cap B) = A \cap B$

Example 2. Use Venn diagrams to illustrate that $(A \cap B')' \cap B = B$.

Solution:



$A \cap B'$



$(A \cap B')' \cap B = B$

EXERCISE 1.4

Illustrate the following identities by means of Venn diagrams.

1. $(A \cap B)' = A' \cup B'$.
2. $(A \cap B \cap C)' = A' \cup B' \cup C'$.
3. $(A \cap B')' \cup B = A' \cup B$.
4. $(A \cup B')' \cap B = A' \cap B$.
5. $(A')' = A$.
6. $A' \cup (A \cap B) = A' \cup B$.
7. $A' \cap (A \cup B) = A' \cap B$.
8. $(A \cap B') \cup (A' \cap B)' = A \cup B'$.

1.5. Algebra of Sets (Supplementary)

The following equations, most of which we have illustrated by Venn diagrams in the previous sections or exercises, are *identities* for any sets A, B, and C in a common universe U.

(1) (a) $A \cap B = B \cap A$. (b) $A \cup B = B \cup A$.

(2) (a) $(A \cap B) \cap C = A \cap (B \cap C)$. (b) $(A \cup B) \cup C = A \cup (B \cup C)$.

(3) (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(4) (a) $A \cap \phi = \phi$. (b) $A \cup \phi = A$.

(5) (a) $A \cap U = A$. (b) $A \cup U = U$.

(6) (a) $A \cap A' = \phi$. (b) $A \cup A' = U$.

(7) (a) $A \cap A = A$. (b) $A \cup A = A$.

(8) (a) $A \cap (A \cup B) = A$. (b) $A \cup (A \cap B) = A$.

(9) (a) $(A \cap B)' = A' \cup B'$. (b) $(A \cup B)' = A' \cap B'$.

(10) (a) $U' = \phi$. (b) $\phi' = U$.

(11) $(A')' = A$.

(12) $A \subseteq B$ if and only if $A \cap B = A$, and $A \subseteq B$ if and only if $A \cup B = B$.

We may accept these identities as the basic axioms of an algebra of sets and prove more complex results by using them. While we have demonstrated the plausibility of each of these results by using Venn diagrams, we must remember that such a demonstration is not a rigorous proof. We may in fact prove these "axioms" in terms of the verbal definitions of the symbols and operations by using logical connectives and a process called set inclusion. (See Section 1.6.)

If we examine closely the (a) and (b) equations in Identities (1) to (10), we notice important similarities in form. These illustrate the very important *Principle of Duality* in Set Theory. This principle states that any established result involving sets and complements and the operations of union and intersection gives a corresponding dual result by replacing U by ϕ and $\cup$ by $\cap$, and vice versa. For example, in Identity (3),

$$\begin{array}{cccccccccccccc}
\text{(a)} & A & \cap & (B & \cup & C) & = & (A & \cap & B) & \cup & (A & \cap & C) \\
 & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \updownarrow & \\
\text{(b)} & A & \cup & (B & \cap & C) & = & (A & \cup & B) & \cap & (A & \cup & C)
\end{array}$$

or in Identity (6),

$$\begin{array}{cccccc}
\text{(a)} & A & \cap & A' & = & \phi \\
 & & \updownarrow & & & \updownarrow \\
\text{(b)} & A & \cup & A' & = & U
\end{array}$$

We may note also that the analogies to the laws of an algebraic field, as in the real number system, are true only for (1), (2), (3(a)), (4), and (5(a)) when $+$ is the analogue of $\cup$, $\times$ of $\cap$, 0 of ϕ, and 1 of U. For example,

$$A \cap \phi = \phi \longleftrightarrow a \times 0 = 0\,,$$

and

$$A \cup \phi = A \longleftrightarrow a + 0 = a\,.$$

The very important differences from the algebra of a field should also be noted; for example, contrast $A \cap A = A$ with $a \times a = a^2$.

Using the basic identities, more complex identities, which have previously been illustrated by Venn diagrams, may now be proved algebraically. In the first example we shall quote the axiom numbers as authorities for reference but these would normally be omitted.

Example 1. Prove that

$$(A \cap B') \cup (A' \cap B)' = A \cup B'\,.$$

Solution 1:

$$\begin{aligned}(A \cap B') \cup (A' \cap B)' &= (A \cap B') \cup (A \cup B') && \text{(9(a)) and (11)}\\ &= [(A \cap B') \cup A] \cup B' && \text{(2(b))}\\ &= A \cup B'\,. && \text{(8(b))}\end{aligned}$$

Solution 2:

$$\begin{aligned}(A \cap B') \cup (A' \cup B)' &= (A \cap B') \cup (A \cup B') && \text{(9(a)) and (11)}\\ &= A \cup B'\,. && \text{(12) and } (A \cap B' \subseteq A \cup B')\end{aligned}$$

It should be noted that other methods of solution are possible As in geometry and most branches of mathematics, alternative solutions are often, and indeed usually, possible.

Example 2. Simplify the set expression

$$[A \cap (B \cup A')] \cup B\,.$$

Solution:

$$\begin{aligned}[A \cap (B \cup A')] \cup B &= [(A \cap B) \cup (A \cap A')] \cup B && \text{(3(a))}\\ &= [(A \cap B) \cup \phi] \cup B && \text{(6(a))}\\ &= (A \cap B) \cup B && \text{(4(b))}\\ &= B\,. && \text{(8(b))}\end{aligned}$$

Some of the steps in the above solution may be omitted after facility in working with the basic identities has been attained.

EXERCISE 1.5

Prove the following set identities.

1. $(S \cap U)' \cup S' = S'$.
2. $(\phi \cap A) \cup B = B$.
3. $A' \cup (A \cap B) = A' \cup B$.
4. $X \cap Y \cap X' = \phi$.
5. $(A \cap B')' \cup B = A' \cup B$.
6. $(P \cup Q') \cap (P' \cap Q)' = P \cup Q'$.
7. $(M' \cap \phi)' \cap M = M$.
8. State and prove the dual identities in questions (1) to (7).

Simplify the following set expressions.

9. $A \cup A' \cup B$.
10. $(P \cap U)' \cap P$.
11. $(P \cup U)' \cup P$.
12. $(A \cap B) \cup (A' \cup B)$.
13. $[M \cap (N \cup M')] \cup N$.
14. $(A \cap U) \cup A'$.
15. $(A \cup \phi) \cap A'$.
16. $(A \cap B) \cup B \cup [(A \cap B) \cap (C \cup D)]$.

1.6. The Fundamental Identities (Supplementary)

As was stated in Section 1.5, the fundamental identities may be regarded as axioms or we may prove them in terms of the verbal definitions of the symbols and operations using logical connectives and a process called set inclusion.

In using the process of set inclusion we select any arbitrary element x of a set A and if we can prove that x must then be an element of a set B, then $A \subseteq B$. Similarly, if $y \in B$ enables us to prove that $y \in A$, then $B \subseteq A$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

By our definitions we know that if $x \in A \cap B$, then $x \in A$ *and* $x \in B$; also if $y \in A \cup B$, then $y \in A$ *or* $y \in B$. The logical connectives used here are the words *and* and *or*. We remember that *or* is interpreted as the inclusive alternative; y belongs to A or to B or to both.

If $x \in A'$, then $x \notin A$ also follows from the definition of the complement of a given set. The logical connective used here is the word *not*.

Fundamental Identity (11) (Section 1.5) is one of the simpler ones to prove and so we will discuss it first.

Example 1. Prove that

$$(A')' = A\,.$$

Solution: If

$$x \in (A')', \quad \text{then} \quad x \notin A'\,. \tag{1}$$

Since

$$x \notin A', \quad \text{then} \quad x \in A\,. \tag{2}$$

Hence

$$(A')' \subseteq A\,. \tag{3}$$

If

$$x \in A, \quad \text{then} \quad x \notin A'\,. \tag{4}$$

Since

$$x \notin A', \quad \text{then} \quad x \in (A')'\,. \tag{5}$$

Hence

$$A \subseteq (A')'\,. \tag{6}$$

Therefore

$$(A')' = A\,. \tag{7}$$

In lines (1) and (2), we use the definition of complement. In line (3), we use the logical law that, if x is an element of $(A')'$ implies that x must then be an element of A, then $(A')'$ is a subset of A. A similar argument is used in lines (4) to (6). Since lines (1) to (6) show that every element of $(A')'$ is an element of A and also that every element of A is an element of $(A')'$, then A and $(A')'$ must be the same set.

Example 2. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\,.$$

Solution: If

$$x \in A \cap (B \cup C)\,,$$

then

$$x \in A \qquad \text{and} \quad x \in B \cup C\,.$$

That is,

$$\begin{array}{lcl} x \in A & \text{and} & x \in B \text{ or } C\,. \\ x \in A \text{ and } B & \text{or} & x \in A \text{ and } C\,. \\ x \in A \cap B & \text{or} & x \in A \cap C\,. \end{array}$$

Thus

$$x \in (A \cap B) \cup (A \cap C)\,.$$

Hence

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)\,.$$

If

$$x \in (A \cap B) \cup (A \cap C)\,,$$

then

$$x \in A \text{ and } B \qquad \text{or} \qquad x \in A \text{ and } C\,.$$

That is,

$$x \in A \quad \text{and} \quad x \in B \text{ or } C\,.$$
$$x \in A \quad \text{and} \quad x \in B \cup C\,.$$

Thus

$$x \in A \cap (B \cup C)\,.$$

Hence

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)\,.$$

Therefore

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\,.$$

Example 3. Prove that

$$A \cup (A \cap B) = A\,.$$

In this case we will assume that if $A \subseteq B$, then $A \cup B = B$ (Fundamental Identity (12)).

Solution: If

$$x \in A \cap B\,,$$

then

$$x \in A \text{ and } B;$$

that is,

$$x \in A\,.$$

Hence

$$(A \cap B) \subseteq A\,.$$

Therefore

$$(A \cap B) \cup A = A\,. \tag{12}$$

Example 4. Prove that

$$\phi' = U\,.$$

In this case we must remember that all elements are elements of the universe under discussion and no $x \in \phi$.

Solution: If

$$x \in \phi'\,,$$

then

$$x \notin \phi\,.$$

Thus

$$x \in U\,.$$

Therefore

$$\phi' \subseteq U\,.$$

If

$$x \in U\,,$$

then

$$x \notin \phi\,.$$

Thus

$$x \in \phi' .$$

Hence

$$U \subseteq \phi' .$$

Therefore

$$\phi' = U .$$

EXERCISE 1.6

1. Using the method of set inclusion prove each of the fundamental identities in Section 1.5 (page 14).

1.7. Cartesian Product

We recall that the set

$$\{(x, y) \mid x \in S,\ y \in T\}$$

is the set of all ordered pairs (x, y) that can be formed by choosing x from a set S and y from a set T.

Example 1. If $S = \{0, 1, 2\}$ and $T = \{4, 5\}$, then

$$\{(x, y) \mid x \in S,\ y \in T\}$$

is the set of ordered pairs

$$\{(0, 4), (0, 5), (1, 4), (1, 5), (2, 4), (2, 5)\} .$$

This set of ordered pairs is called the Cartesian product set $S \times T$ (read S cross T). We note that the Cartesian product set $T \times S$ or $\{(x, y) \mid x \in T,\ y \in S\}$ is *not* the same set of ordered pairs although it will contain the same number of elements. The set S contains three members and the set T, two members. Both $S \times T$ and $T \times S$ contain six members, each member being an ordered pair of numbers.

$$T \times S = \{(4, 0), (5, 0), (4, 1), (5, 1), (4, 2), (5, 2)\} .$$

We know from Cartesian plane geometry that there is a one-to-one correspondence between the elements of the set $Re \times Re$ or $\{(x, y) \mid x \in Re,\ y \in Re\}$ and the points of the Cartesian plane.

From this example, it would appear that if a set A contains r elements and a set B contains s elements, then the set $A \times B$ will contain rs elements. A lattice representation of $A \times B$ can be formed easily if A and B are finite sets with a reasonably small number of elements.

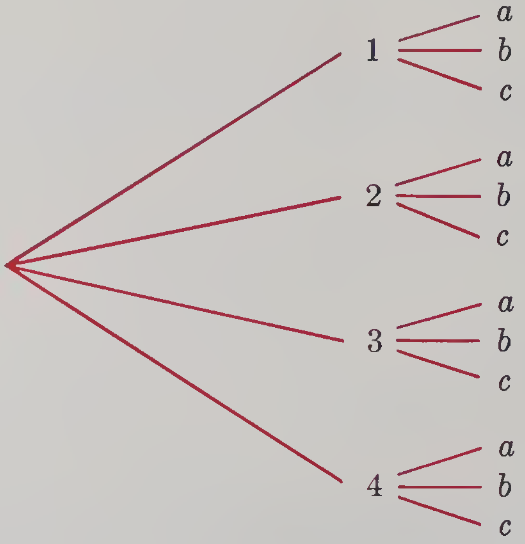
Example 2. Find the number of elements in $A \times B$ if $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.

Solution:

B A	a	b	c
1	$(1, a)$	$(1, b)$	$(1, c)$
2	$(2, a)$	$(2, b)$	$(2, c)$
3	$(3, a)$	$(3, b)$	$(3, c)$
4	$(4, a)$	$(4, b)$	$(4, c)$

From this we see that $A \times B$ contains 12 elements. A contains 4 elements, B contains 3 elements, and $A \times B$ contains 4×3 elements.

Another way of observing the number of elements in $A \times B$ is by means of a tree diagram.



We note that the first member of each ordered pair has associated with it each element of the second set as a second member of an ordered pair. In the example, A contains 4 elements; each of these elements is paired in turn with each element of the set B and thus three ordered pairs are formed with each of the 4 elements of A as a first member. The total number of elements is, therefore, $4 \times 3 = 12$.

If $A = \{2, 3, 5\}$, the Cartesian product set $A \times A$ will contain $3 \times 3 = 9$ elements. If we are asked for the subset of $A \times A$ which has as elements the ordered pairs which pair only *distinct* elements of A, the elements $(2, 2)$, $(3, 3)$, and $(5, 5)$ are not needed. From the first member 2, we can form two ordered pairs $(2, 3)$ and $(2, 5)$, and similarly for each of the three first members. The number of elements in the subset is $3 \times 2 = 6$. There are 3 choices for the first member and, for *each* of these choices, there are 2 members for the second choice.

EXERCISE 1.7

1. If $A = \{a, b, c, d\}$ and $B = \{r, s\}$, list the elements of $A \times B$ and $B \times A$.
2. If $P = \{1, 2, 3, 4, 5\}$, list the elements of $P \times P$.
3. If $P = \{1, 2, 3, 4, 5\}$, list the elements of the subset of $P \times P$ that consists of all ordered pairs in $P \times P$ in which the first member is not equal to the second member.
4. In questions (2) and (3), state the number of elements in the sets listed.
5. Find

$$\{(x, y) \mid x \in S, y \in T\}$$

if

(a) $S = \{-2, -1, 0, 1, 2\}$, $T = \{0, 1\}$;
(b) $S = \{1, 2, 3\}$, $T = \{0\}$;
(c) $S = \{0\}$, $T = \{y \in N \mid y < 10\}$;
(d) $S = \{x \in N \mid x < 12 \text{ and } x \text{ is prime}\}$, $T = \{x \in N \mid 4x \leq 12\}$.

6. In question (5), state the number of elements in S, in T, and in $S \times T$.
7. How many elements are in $A \times B$ if $A = \{0, 1, 2, 3\}$ and $B = \phi$?
8. How many elements are in $\phi \times \phi$?
9. If $A = \{-1, 0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1\}$, list the elements in $A \times B \times C$ and $B \times A \times C$.

1.8. Number of Elements in $A \cup B$

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3, 4, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5, 6\}$. We see that the number of elements in A is 5, in B is 4, in $A \cap B$ is 3, and in $A \cup B$ is 6. We will represent the number of elements in a given set as $n(A)$. In the example given, $n(A) = 5$, $n(B) = 4$, $n(A \cap B) = 3$, $n(A \cup B) = 6$.

We notice that $n(A \cup B)$ is *not* equal to $n(A) + n(B)$ but, in this example,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)\,. \qquad (1)$$

In general if we add the number of elements in A to the number of elements in B, we include the common elements in $A \cap B$ twice. In the example, the elements 3, 4, and 5 appear in both A and B, that is, in $A \cap B$, and are counted twice in adding the number of elements of A to the number of elements of B. In $A \cup B$ the common elements are counted once only and so the result in (1) is true in general.

Example 1. If $A = \{a, b, c, d, e, f\}$ and $B = \{e, f, g, h\}$, find the number of elements in $A \cap B$ and in $A \cup B$.

Solution:

$$\begin{aligned} A \cap B &= \{e, f\} \\ n(A \cap B) &= 2\,. \\ n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 6 + 4 - 2 \\ &= 8\,. \end{aligned}$$

EXERCISE 1.8

1. In each of the following examples, state the values of $n(A \times B)$, $n(A \cap B)$, and $n(A \cup B)$.
   (a) $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 2, 3, 4\}$
   (b) $A = \{a, b, c, d\}$, $B = \{e, f, g\}$
   (c) $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$
   (d) $A = \{x \in N \mid x \leq 12, x \text{ is prime}\}$, $B = \{x \in N \mid x \leq 12, x \text{ is even}\}$
   (e) $A = \{x \in N \mid x < 10\}$, $B = \phi$
2. If P and Q are disjoint sets, give a formula for $n(P \cup Q)$ and $n(P \cap Q)$.
3. If $T \subset S$, what is the value of $n(T \cap S)$ and $n(T \cup S)$?
4. If $n(A \cup B) = n(A)$, what is the relationship between sets A and B?
5. If $n(R \cup S) = n(R) + n(S)$, what is the value of $n(R \cap S)$?
6. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 5\}$, state the value of $n(A \cup B \cup C)$.

1.9. A×B×C and A∪B∪C

The ideas of the previous two sections can be extended to include more than two sets. For example,

$$A \times B \times C = \{(x, y, z) \mid x \in A, y \in B, z \in C\}\,.$$

The set $A \times B \times C$ is a set of ordered triples, the first member of each triple being an element of set A, the second member, an element of set B, and the third member, an element of set C. Again there is a one-to-one correspondence between the elements of the set $\{(x, y, z) \mid x, y, z \in Re\}$ and the points of Cartesian three-dimensional space.

The set

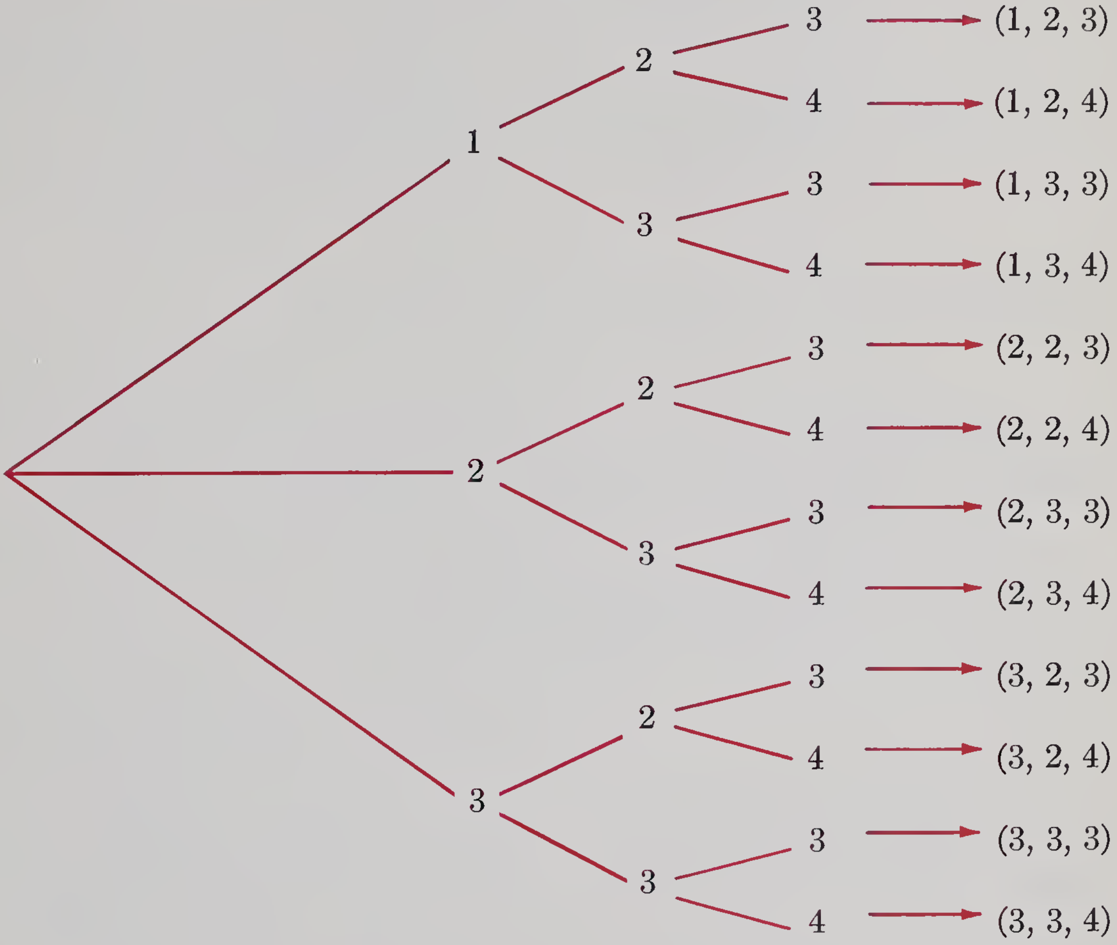
$$A \times B \times C \times D = \{(x, y, z, w) \mid x \in A, y \in B, z \in C, w \in D\}$$

extends the idea to four sets. Is there a correspondence between the set $\{(x, y, z, w) \mid x, y, z, w \in Re\}$ and the points in any geometrical space?

Example 1. $A = \{1, 2, 3\}$, $B = \{2, 3\}$, $C = \{3, 4\}$.

$$\begin{aligned} A \times B \times C = \{&(1, 2, 3), (1, 2, 4), (1, 3, 3), (1, 3, 4), (2, 2, 3), (2, 2, 4), \\ &(2, 3, 3), (2, 3, 4), (3, 2, 3), (3, 2, 4), (3, 3, 3), (3, 3, 4)\}. \end{aligned}$$

In this example, $n(A) = 3$, $n(B) = 2$, $n(C) = 2$ and $n(A \times B \times C) = 3 \times 2 \times 2 = 12$. Again, a tree diagram may prove helpful in setting up the ordered triples.



We know that if $n(A) = r$, $n(B) = s$, then $n(A \times B) = rs$. It would appear that there is a natural extension of this to any number of sets and that if

$$n(A_1) = a_1, \quad n(A_2) = a_2, \quad \cdots, \quad n(A_n) = a_n,$$

then

$$n(A_1 \times A_2 \times \cdots \times A_n) = a_1 a_2 \cdots a_n$$

Returning to the example where $n(A) = 3$, $n(B) = 2$, and $n(C) = 2$, we see that $n(A \times B) = 6$ as found in Section 1.7. Each of these 6 ordered pairs can, in turn, be paired with one element of C to produce one element of $A \times B \times C$. The 6 ordered pairs of $A \times B$ produce 6 ordered triples with the element 3 of C and a further 6 ordered triples with the element 4 of C. The total number of elements in $A \times B \times C$ is, therefore, $6 \times 2 = 12$.

Knowing that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ and using the identities established in the algebra of sets, we can calculate the number of elements in $A \cup B \cup C$.

$$\begin{aligned}
n(A \cup B \cup C) &= n[A \cup (B \cup C)] \\
&= n(A) + n(B \cup C) - n[A \cap (B \cup C)] \\
&= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)] \\
&= n(A) + n(B) + n(C) - n(B \cap C) - \\
&\quad [n(A \cap B) + n(A \cap C) - n((A \cap B) \cap (A \cap C))] \\
&= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) \\
&\quad + n(A \cap B \cap C) .
\end{aligned}$$

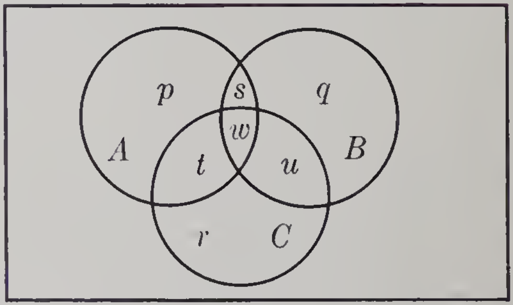
Note that in the last line we have used the result that

$$(A \cap B) \cap (A \cap C) = A \cap A \cap B \cap C = A \cap B \cap C .$$

This formula appears to be rather complicated but, by examining a Venn diagram, we should be able to interpret it more easily.

In the Venn diagram, we have indicated by the letters, p, q, r, s, t, u, w, the number of elements in each of the subsets of $A \cup B \cup C$. We see that

$$\begin{aligned}
n(A \cup B \cup C) &= p + q + r + s + t + u + w \\
n(A) &= p + s + t + w , \\
n(B) &= q + s + u + w , \\
n(C) &= r + t + u + w .
\end{aligned}$$



Thus

$$\begin{aligned}
n(A) + n(B) + n(C) &= p + q + r + 2s + 2t + 2u + 3w \\
&= n(A \cup B \cup C) + s + t + u + 2w . \\
n(A \cap B) &= s + w , \\
n(B \cap C) &= u + w , \\
n(A \cap C) &= t + w , \\
n(A \cap B \cap C) &= w \\
n(A \cap B) + n(B \cap C) + n(A \cap C) &= s + t + u + 3w .
\end{aligned}$$

Therefore

$$\begin{aligned} n(A\cup B\cup C) &= n(A)+n(B)+n(C)-(s+t+u)-2w \\ &= n(A)+n(B)+n(C)-(s+t+u+3w)+w \\ &= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C) \\ &\quad -n(A\cap C)+n(A\cap B\cap C)\,. \end{aligned}$$

Example 2. If $A=\{1, 2, 3, 4, 5\}$, $B=\{3, 4, 5, 6\}$, $C=\{5, 6, 7,\}$ find

$$n(A\times B\times C) \quad \text{and} \quad n(A\cup B\cup C)\,.$$

Solution:

$$n(A\times B\times C)=5\times 4\times 3=60\,.$$

$$\begin{aligned} n(A\cup B\cup C) &= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C) \\ &\quad +n(A\cap B\cap C) \\ &= 5+4+3-3-2-1+1 \\ &= 7\,. \end{aligned}$$

EXERCISE 1.9

1. If $A=\{a, b, c, d\}$, $B=\{b, c, d, e\}$, $C=\{c, d, e, f, g\}$, find the number of elements in $A\times B\times C$ and in $A\cup B\cup C$.
2. For the following sets P, Q, R, find $n(P\times Q\times R)$ and $n(P\cup Q\cup R)$.
   (a) $P=\{-2, -1, 0, 1, 2\}$, $Q=\{0, 1, 2, 3\}$, $R=\{2, 3\}$
   (b) $P=\{1, 2, 3\}$, $Q=\{1, 2, 3, 4\}$, $R=\{1, 2, 3, 4, 5\}$
   (c) $P=\{a, b, c\}$, $Q=\{d, e, f\}$, $R=\{g, h\}$
   (d) $P=\{1, 2, 3, 4\}$, $Q=\{0\}$, $R=\{0, 1, 2\}$
   (e) $P=\{0, 5, 10\}$, $Q=\{10\}$, $R=\phi$
3. If $A=\{1, 2, 3, 4\}$, what is the value of $n(A\times A\times A)$ and $n(A\cup A\cup A)$?
4. If $n(A)=2$, $n(B)=3$, $n(C)=5$, and $A\subset B\subset C$, find the value of $n(A\cup B\cup C)$.
5. If $A=\{1, 2, 3, 4\}$, find the number of elements in the subset of $A\times A\times A$ that has three distinct members in each ordered triple.
6. If $n(A)=8$, find (i) $n(A\times A\times A)$ and (ii) the number of elements in $A\times A\times A$ that have three distinct members in each ordered triple.
7. If P, Q, and R are disjoint sets, state a formula for $n(P\cup Q\cup R)$.
8. If $n(A)=6$, what is the value of $n(A\times A\times A)$ and $n(A\cup A\cup A)$?
9. If $A=\{0, 1, 2\}$, $B=\{-1, 0\}$, $C=\{2, 3\}$, $D=\{1, 2, 3\}$, calculate the value of $n(A\cup B\cup C\cup D)$.

1.10. Statistics of Attributes (Supplementary)

One interesting application of the previous two sections is in checking or analysing certain statistical data. In many cases a Venn diagram may be sufficient to solve the problem, while in other cases the formulae developed in Section 1.9 may have to be used.

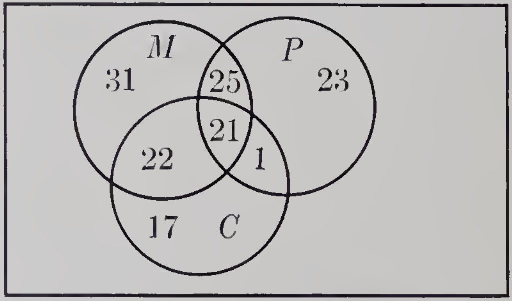
Example 1. In a certain high school there are 187 students in Grade 13. Of these students

99 study mathematics,
70 study physics,
61 study chemistry,
46 study both mathematics and physics,
22 study both physics and chemistry,
43 study both mathematics and chemistry,
21 study mathematics, chemistry, and physics.

How many students study at least one of these subjects?

Solution: Let

$$M = \{\text{students of mathematics}\},$$
$$P = \{\text{students of physics}\},$$
$$C = \{\text{students of chemistry}\}.$$



Since 21 students study all three subjects, we can enter in the number 21 in the section representing $M \cap P \cap C$ in the Venn diagram.

46 students study mathematics and physics; so that will be the number in $M \cap P$. 21 of these, however, are already entered in $M \cap P \cap C$ leaving 25 to be entered in the remaining portion of $M \cap P$, that is, in $M \cap P \cap C'$. Similarly, the numbers 22 and 1 can be entered in $M \cap C \cap P'$ and $C \cap P \cap M'$, respectively.

On examining the diagram, we see that we have now accounted for 68 students who study mathematics along with either or both of physics and chemistry. Since there are 99 mathematics students in all, the remaining 31 must study mathematics only and that number can be entered in the section representing $M \cap P' \cap C'$. Similarly the numbers 17 and 23 can be entered in the sections representing $C \cap M' \cap P'$ and $P \cap C' \cap M'$, respectively.

When we now total the numbers in the seven regions, we obtain a total of 140 students who study at least one of the three subjects. Therefore

$$M \cup P \cup C = 140\,.$$

Alternatively, from the given data we know that

$$n(M) = 99,\ n(\mathrm{P}) = 70,\ n(C) = 61,$$
$$n(M \cap P) = 46,\ n(P \cap C) = 22,\ n(M \cap C) = 43\,,$$
$$n(M \cap P \cap C) = 21\,.$$

Therefore

$$\begin{aligned} n(M \cup P \cup C) &= n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) \\ &\quad - n(M \cap C) + n(M \cap P \cap C) \\ &= 99 + 70 + 61 - 46 - 22 - 43 + 21 \\ &= 251 - 111 \\ &= 140\,. \end{aligned}$$

Therefore 140 students study at least one of the subjects.

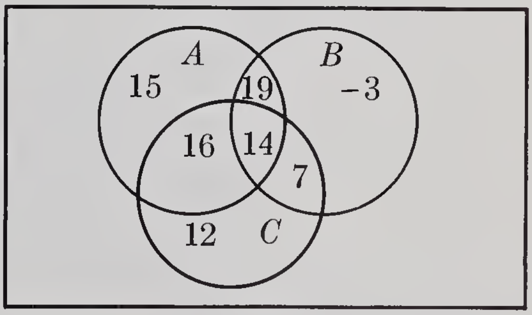
Example 2. In a certain factory the following data were compiled from a survey of employees.

64 were Canadian born.
37 were married.
49 were male.
33 of the Canadians were married.
21 of the males were married.
30 of the Canadian born were male.
14 were married Canadian born males.

Analyse the consistency of the data.

Solution: Let

$$A = \{\text{Canadian born}\},$$
$$B = \{\text{married}\},$$
$$C = \{\text{males}\}.$$



When we enter the numbers in the various regions of the Venn diagram, we see that $n(B \cap A' \cap C') = -3$. This is impossible since it symbolizes the statement that the number of non-Canadian married females is -3. We conclude, therefore, that the data are inconsistent and that the compiler was in error.

EXERCISE 1.10

1. In a survey of 82 people the following information was obtained: 41 of those interviewed were male, 42 had voted Liberal at the last election, 38 were married, 12 of the males were married, 21 of the males had voted Liberal, 10 of the Liberal voters were married, and 4 of the Liberal voters were married males. How many unmarried females had not voted Liberal?

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2. In an advertisement, a firm which produced three products A, B, and C stated that, of 10,000 families surveyed,

   6974 used product A,
   5417 used product B,
   7246 used product C,
   2950 used both A and B,
   4623 used both A and C,
   3545 used both B and C, and
   1720 used all three products.

   Why should the firm be prosecuted for fraudulent advertising?

3. Data obtained from a survey of 200 students were reported as follows.

   114 studied Mathematics.
   100 studied History.
   67 studied French.
   48 studied Mathematics and History.
   41 studied Mathematics and French.
   37 studied History and French.
   17 studied all three.

   How many students studied
   (a) exactly two of these subjects?
   (b) exactly one of these subjects?
   (c) none of these subjects?

4. From 600 scientists interviewed, the following data were obtained.

   339 had studied physics.
   345 had studied chemistry.
   315 had studied biology.
   151 had studied physics and chemistry.
   157 had studied physics and biology.
   175 had studied chemistry and biology.

   Assuming that all 600 had studied at least one of these subjects, how many had studied all three?

5. Of 1200 students in Dudgeon High School, 900 attended the first football game. The team lost 54-0 and only 400 attended the second game.
   (a) What is the minimum number that could have seen both games?
   (b) If 250 attended both games, how many did not attend either game?

6. A dime, a nickel, and a quarter were tossed together 100 times and turned up heads 54, 47, and 49 times, respectively. The dime and nickel were heads together 25 times; the dime and quarter were heads together 27 times; the nickel and quarter were heads together 22 times. If all 3 were tails 11 times, how many times were all 3 heads?

Chapter Summary

$x \in A$	x is an element of set A.
$A \subseteq B$	A is a subset of B.
$A \cap B$	The set of all elements in both A and B
$A \cup B$	The set of all elements in A or B or both
A'	The complement of A

Fundamental Identities

(1) (a) $A \cap B = B \cap A$. (b) $A \cup B = B \cup A$.
(2) (a) $(A \cap B) \cap C = A \cap (B \cap C)$. (b) $(A \cup B) \cup C = A \cup (B \cup C)$.
(3) (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(4) (a) $A \cap \phi = \phi$. (b) $A \cup \phi = A$.
(5) (a) $A \cap U = A$. (b) $A \cup U = U$.
(6) (a) $A \cap A' = \phi$. (b) $A \cup A' = U$.
(7) (a) $A \cap A = A$. (b) $A \cup A = A$.
(8) (a) $A \cap (A \cup B) = A$. (b) $A \cup (A \cap B) = A$.
(9) (a) $(A \cap B)' = A' \cup B'$. (b) $(A \cup B)' = A' \cap B'$.
(10) (a) $U' = \phi$. (b) $\phi' = U$.
(11) $(A')' = A$.
(12) $A \subseteq B$ iff $A \cap B = A$ and $A \subseteq B$ iff $A \cup B = B$.

$$S \times T = \{(x, y \mid x \in S, y \in T\}.$$
$$n(S \times T) = n(S) \times n(T).$$
$$n(S \cup T) = n(S) + n(T) - n(S \cap T).$$
$$n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T) - n(S \cap R) - n(R \cap T) + n(S \cap T \cap R).$$

REVIEW EXERCISE 1

1. If $A = \{2, 3, 5, 7, 11, 13\}$, $B = \{1, 3, 5, 7, 11, 13\}$, list the elements of the sets $A \cap B$, $A \cup B$. List the elements of set $C = \{x \in B \mid x \text{ is a prime}\}$.

2. If $U = \{x \in N \mid x < 20\}$, $A = \{x \in U \mid x \text{ is a prime}\}$, $B = \{x \in U \mid x \text{ is odd}\}$, list the elements of

$A \cap B$, $A \cup B$, A', B', $(A \cap B)'$, $(A \cup B)'$, $A \cap B'$.

3. Construct Venn diagrams to illustrate the following sets.

(a) $A \cap B'$ (b) $A' \cap B$
(c) $A \cap B$ when $A \subseteq B$ (d) $A \cup B$ when $A \subseteq B$

(e) $A \cup A' \cup B$ (f) $(A \cap B) \cap (A' \cap B)$
(g) $(P \cap Q')' \cup Q$ (h) $A \cap B \cap C'$
(i) $(P \cap Q) \cup R$ (j) $A' \cap B' \cap C'$

4. Use the fundamental identities to simplify the following set expressions.
(a) $A \cup A' \cup B$ (b) $(C \cap D) \cup (C' \cap D)$
(c) $(P \cup Q) \cap P'$ (d) $[(C \cap \phi) \cup C]'$
(e) $(R \cup S) \cup R \cup [(R \cup S) \cap (P \cup Q)]$

5. $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$, $C = \{5, 6, 7\}$
(a) State which of the following are true.
(i) $A \subseteq B$. (ii) $C \subseteq B$. (iii) $A \cap B = A$.
(iv) $A \cap B = \phi$. (v) $A \cap C = \phi$. (vi) $A \cup C = B$.
(b) Draw a Venn diagram to illustrate the relationship between the sets A, B, and C.

6. If $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,6\}$, $C = \{5,6,7\}$, find the value of each of the following.
(a) $n(A \times C)$ (b) $n(A \times B \times C)$
(c) $n(A \cap B)$ (d) $n(A \cup C)$
(e) $n(B \cap C)$ (f) $n(A \cup B \cup C)$

7. If $A = \{1,2,3,4\}$, find the number of elements in the subset of $A \times A$ for which the first and second members of each ordered pair are unlike.

Supplementary.

8. Prove, by using set inclusion, that
(a) $A \cup A' \cup B = U$, (b) $(P \cap Q') \cup Q = P \cup Q$,
(c) $X \cap Y \cap X' = \phi$, (d) $(M' \cap \phi)' \cap M = M$.

9. In a class of 36 students, each of whom must study French or Latin, 31 study French and 15 study both French and Latin. How many study Latin?

10. A display was given by the gymnastic club of a certain school. 94 visitors attended on the first day, 123 on the second day, and 122 on the third day. 47 attended on both the first and second days, 38 on both the first and third days, and 45 on both the second and third days. If 22 visitors attended on all three days, how many different visitors saw the display?

Chapter 2

ARRANGEMENTS AND SUBSETS

2.1. Arrangements of a Set

If two sets A and B contain r elements and s elements, respectively, we know from Chapter 1 that the Cartesian product set $A \times B$ contains rs elements. This could be reworded to state that, if a certain act can be performed in r different ways and if, for each of these ways, a second act can be performed in s different ways, then the two acts can be performed successively in rs different ways.

Consider the set of five elements $A = \{a, b, c, d, e\}$. The number of elements in the subset of $A \times A \times A$, in which no two of the members in each ordered triple are like, is $5 \times 4 \times 3$ or 60. We can arrive at this by considering that we have 3 spaces $\boxed{}\boxed{}\boxed{}$ to fill for each element of $A \times A \times A$. The first space can be filled with any one of the five letters. For each of these ways the second space can be filled with any one of the four remaining letters. Hence the first two spaces can be filled in $5 \times 4 = 20$ different ways. For *each* of these ways, the third space can be filled with any of the three remaining letters. The three spaces can, therefore, be filled in $5 \times 4 \times 3$ or 60 different ways.

In this case we have sixty 3-arrangements of five objects. We note that the set $A \times A \times A$ contains $5 \times 5 \times 5$ or 125 elements, each of which is an ordered triple, and also includes ordered triples with two or three like letters.

This can be extended to include arrangements of any finite number of elements from any finite set.

DEFINITION. An r-arrangement of n objects is an *ordered* selection of r of the objects $(r \in W, n \in N)$.

The number of r-arrangements may be deduced in the same way as we found the number of 3-arrangements of five objects.

n	$n - 1$	$n - 2$	$\cdots$	$n - (r - 1)$

We may select the first object in n different ways. For *each* of these ways, the second object can be selected from any one of the remaining $(n - 1)$ objects.

The first two objects can, therefore, be selected in $n(n-1)$ different ways and, for *each* of these ways, the third object can be selected in $(n-2)$ different ways. The first three objects can be selected in $n(n-1)(n-2)$ ways. Continuing this procedure, the rth object can be selected in $n-(r-1)$ different ways $(r \leq n)$. The r objects can be selected in $n(n-1)(n-2)\cdots(n-r+1)$ different ways.

Therefore, the number of r-arrangements of n objects is

$$n(n-1)(n-2)\cdots(n-r+1) \qquad (r \leq n)\,.$$

Various symbols are used to indicate the number of r-arrangements of n objects taken r at a time. The one we shall use in this text is $n_{(r)}$. Others used are $P(n, r)$, $P\binom{n}{r}$, and ${}_nP_r$. The number of arrangements of five objects taken three at a time is $5_{(3)}$ or $5 \times 4 \times 3$.

Example 1. Find the number of 4-arrangements of six objects.

Solution: Number of 4-arrangements

$$\begin{aligned} &= 6_{(4)} \\ &= 6 \times 5 \times 4 \times 3 \\ &= 360\,. \end{aligned}$$

Example 2. How many different licence plates, each containing five digits, can be made using the ten digits 0 to 9 if

(a) the first digit cannot be 0 and repetition of digits is not allowed?

(b) the first digit cannot be 0 but repetitions are allowed?

In how many arrangements in (b) do repetitions actually occur?

Solution:

(a) Since the first digit cannot be 0, there are nine choices for the first digit. The other digits form 4-arrangements of the nine remaining digits. Therefore

$$\begin{aligned} \text{number of licence plates} &= 9 \times 9_{(4)} \\ &= 9 \times 9 \times 8 \times 7 \times 6 \\ &= 27{,}216\,. \end{aligned}$$

(b) If repetitions are allowed, there are ten choices for each digit after the first. Therefore

$$\begin{aligned} \text{number of licence plates} &= 9 \times 10^4 \\ &= 9 \times 10{,}000 \\ &= 90{,}000\,. \end{aligned}$$

The number of arrangements in which repetitions occur is

$$\begin{aligned} &90{,}000 - 27{,}216 \\ &= 62{,}784\,. \end{aligned}$$

Example 3. How many arrangements of all of the letters of the word *numbers* are possible if

(a) there is no restriction?

(b) the letter s must be last?

(c) the vowels must be together?

Solution:

(a) Number of arrangements with no restrictions

$$\begin{aligned} &= 7_{(7)} \\ &= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 5040\,. \end{aligned}$$

(b) Since s must be placed last, the other six letters may be arranged in $6_{(6)}$ ways. Therefore

number of arrangements with s last

$$\begin{aligned} &= 6_{(6)} \\ &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720\,. \end{aligned}$$

(c) Since the vowels must be together, they can be considered as one object. Therefore the number of arrangements is $6_{(6)}$. But for *each* of these arrangements the u and e may be interchanged without altering the position of the other letters. Therefore

total number of arrangements with vowels together

$$\begin{aligned} &= 2 \times 6_{(6)} \\ &= 2 \times 720 \\ &= 1440\,. \end{aligned}$$

EXERCISE 2.1

1. Find the value of

   (i) $4_{(2)}$, (ii) $16_{(2)}$, (iii) $7_{(4)}$,
   (iv) $15_{(3)}$, (v) $5_{(5)}$, (vi) $6_{(4)}$.

2. Find the number of 3-arrangements of eight objects.

3. Find the number of 2-arrangements of seven objects.

4. Find the number of 5-arrangements of five objects.

5. Find the number of 6-arrangements of six objects.

6. How many three-digit numbers can be made from the digits 1, 2, 3, 4, 5 if repetitions are not allowed?

7. How many three-digit numbers can be made from the digits 0, 1, 2, 3, 4, if repetitions are not allowed?

8. How many three-digit numbers can be made from the digits 0, 1, 2, 3, 4, if repetitions are allowed?

9. If there are five possible routes by which a driver can travel from Toronto to Ottawa and three possible routes from Ottawa to Montreal, how many different ways are there to travel from Toronto to Montreal via Ottawa?

10. If a dime, a nickel, and a quarter are tossed together, in how many different ways can they fall?

11. In how many ways can eight boys be arranged in a row if
    (a) there is no restriction?
    (b) two boys, John and Jim, must be together?
    (c) John and Jim must be kept apart?

12. How many batting orders are possible for a baseball team of nine players if the pitcher must bat in ninth position?

13. How many possible signals can be made using five different signal flags arranged one above the other?

14. In how many ways can a chairman, a vice-chairman, and a secretary be chosen from a committee of nine members?

15. There are four routes between two towns. In how many ways can a driver travel by one route and return by a different route?

16. In how many ways can the letters of the word *factor* be arranged if
    (a) there is no restriction?
    (b) the first letter must be a consonant?
    (c) the second and fifth letters must be vowels?

17. The dial of a combination lock has one hundred different numbers on it. The lock is opened by dialing three different numbers in a particular order. How many different combinations can be produced?

18. There are ten teams in a hockey league. How many games are in the schedule if each team must play each of the other teams once at home and once away from home?

19. There are six teams in another hockey league. Each team must play each of the other teams seven times on home ice and seven times away from home. How many games are played each season?

2.2. Factorials

In Section 2.1 we have seen that the number of 3-arrangements of five objects is

$$5_{(3)} = 5 \times 4 \times 3 ,$$

and that the number of 5-arrangements of five objects is

$$5_{(5)} = 5 \times 4 \times 3 \times 2 \times 1 .$$

In the latter case we have the product of all the natural numbers from 1 to 5 inclusive. Such a product is written as 5! and read as "five factorial". We may understand that

$$n_{(n)} = n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

is the product of all the natural numbers from 1 to n inclusive. (Note that we must assume that n is a natural number.)

Probably the most rigorous way to define factorial notation is by a recursive definition,

$$\begin{aligned} 0! &= 1, \\ (n+1)! &= n!(n+1) \qquad (n \geq 0). \end{aligned}$$

It may appear strange to define 0! as 1, but the reason for this definition will appear later.

From the recursive definition, we obtain

$$\begin{aligned} 1! &= 0!(1) = 1 , \\ 2! &= 1!(2) = 1 \times 2 , \\ 3! &= 2!(3) = 1 \times 2 \times 3 , \\ 4! &= 3!(4) = 1 \times 2 \times 3 \times 4 , \end{aligned}$$

and so on, in agreement with our more intuitive concept of the product of all the natural numbers from 1 to n inclusive.

The number of r-arrangements of n elements may be expressed in terms of factorials. We know that

$$\begin{aligned} 5_{(3)} = 5 \times 4 \times 3 &= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \\ &= \frac{5!}{2!} . \end{aligned}$$

$$\begin{aligned} 16_{(4)} &= 16 \times 15 \times 14 \times 13 \\ &= \frac{16 \times 15 \times 14 \times 13 \times 12!}{12!} \\ &= \frac{16!}{12!} . \end{aligned}$$

Similarly,

$$\begin{aligned} n_{(r)} &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}. \end{aligned}$$

Therefore

$$n_{(r)} = \frac{n!}{(n-r)!}.$$

Note: This implies that

$$\begin{aligned} n_{(n)} &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!}. \end{aligned}$$

But we know that

$$n_{(n)} = \mathrm{n}!.$$

That is,

$$\frac{n!}{0!} = \mathrm{n}!.$$

This will only be true if $0!$ is equal to 1, which agrees with the definition that $0! = 1$. With any other definition we would lose consistency.

Example 1. Evaluate $\frac{52!}{50!}$

Solution:

$$\begin{aligned} \frac{52!}{50!} &= \frac{52\cdot 51\cdot 50!}{50!} \\ &= 52\cdot 51 \\ &= 2652. \end{aligned}$$

Example 2. Solve for n in $\frac{(n+2)!}{n!} = 56.$

Solution:

$$\begin{aligned} \frac{(n+2)!}{n!} &= 56. \\ \frac{(n+2)(n+1)n!}{n!} &= 56. \\ (n+2)(n+1) &= 56. \\ n^2+3n+2 &= 56. \\ n^2+3n-54 &= 0. \\ (n+9)(n-6) &= 0. \end{aligned}$$

Therefore

$$n = 6 \qquad (n \in N).$$

Example 3. Prove that $(n+1)_{(r+1)} = (n+1)n_{(r)}$.

Solution:

$$\begin{aligned}(n+1)_{(r+1)} &= \frac{(n+1)!}{(n+1-r-1)!}\\ &= \frac{(n+1)n!}{(n-r)!}\\ &= (n+1)\frac{n!}{(n-r)!}\\ &= (n+1)n_{(r)}\,.\end{aligned}$$

Therefore

$$(n+1)_{(r+1)} = (n+1)n_{(r)}\,.$$

EXERCISE 2.2

In each of the following questions,

$$n, r \in N, \quad n \geq r\,.$$

1. Find the value of each of the following.

(a) $\frac{8!}{6!}$ (b) $\frac{6!}{3!3!}$ (c) $\frac{20!}{19!}$

(d) $\frac{8!}{5!3!}$ (e) $\frac{7!}{5!2!} + \frac{7!}{3!4!}$ (f) $\frac{20!}{17!3!}$

2. Express each of the following as a single factorial expression.

(a) $(n+1)n!$ (b) $\frac{(n+7)!}{n+7}$ (c) $(n-r+1)(n-r)!$

3. Simplify the following as far as possible.

(a) $\frac{(n+5)!}{(n+3)!}$ (b) $\frac{n!}{(n-1)!}$

(c) $\frac{(n+1)!}{n!}$ (d) $\frac{(n-r+1)!}{(n-r)!}$

(e) $\frac{(n-r)!}{(n-r-1)!}$ (f) $\frac{(n-r+1)!}{(n-r-1)!}$

4. Solve the following equations.

(a) $\frac{(n+5)!}{(n+4)!} = 7.$ (b) $\frac{(n+2)!}{n!} = 20.$

(c) $n_{(2)} = 30.$ (d) $\frac{n!}{2(n-2)!} = 6.$

(e) $\frac{(n+1)!}{(n-1)!} = 12.$ (f) $\frac{(n-1)!}{(n-3)!} = 42.$

5. Prove that $n_{(r+1)} = (n-r)n_{(r)}$.

2.3. Arrangements with Like Elements

How many six-digit numbers can be made using the digits 1, 1, 1, 2, 2, 3? If the digits were all different, we know that the number of numbers would be 6!. In this case, however, certain of the digits are alike and interchanging the positions of the three 1's or the two 2's among themselves without altering the positions of the other digits would not produce another number. The number of arrangements must be less than 6!. Our problem is to discover the number of distinct arrangements possible. To do this we imagine some means of distinguishing among the like objects. We may, for instance, temporarily label the 1's and 2's with distinctive subscripts 1_1, 1_2, 1_3, 2_1, 2_2. In this case $1_1\ 1_2\ 1_3\ 2_2\ 3\ 2_1$ would be a different arrangement from $1_2\ 1_1\ 1_3\ 2_2\ 3\ 2_1$. The three 1's can then be rearranged among themselves in 3! ways without altering the position of the other digits. Similarly, the two 2's can be rearranged in 2! ways without altering the positions of the other digits. We shall assume that there are x arrangements of the six digits taken all at a time with three alike of one kind (the 1's) and two alike of another (the 2's).

Let the number of arrangements be x. If the three 1's were unlike, then, for *each* of these x arrangements, the 1's could be rearranged among themselves in 3! ways without altering the positions of the other digits.

Therefore the number of arrangements with unlike 1's would be $x \cdot 3!$. Similarly, if the two 2's were unlike, they could be rearranged in 2! ways without altering the positions of the other digits.

Therefore the number of arrangements with unlike 1's and 2's would be $x \cdot 3! \cdot 2!$.

But if the 1's and 2's were unlike, the number of arrangements would be 6!. Hence

$$x \cdot 3! \cdot 2! = 6!.$$

$$x = \frac{6!}{3!2!}$$

$$= 60.$$

Therefore the number of six-digit numbers is 60.

This process can obviously be extended to arrangements of any number of objects taken *all* at a time with certain of the objects alike.

To Find the Number of Arrangements of n Objects When Some Are Alike

To find the number of arrangements of n objects taken all at a time with n_1 alike of one kind, n_2 alike of a second, $\cdots$, n_r alike of an rth kind $(n \geq n_1 + n_2 + \cdots + n_r)$, we proceed as follows.

Let the number of arrangements be x. If the n_1 like objects were unlike, then, for each of these x arrangements, the n_1 like objects could be rearranged among themselves in $n_1!$ ways without altering the positions of the other objects.

Therefore the number of arrangements would be $x \cdot n_1!$.

Similarly, if the n_2 like objects were unlike, each of these $x \cdot n_1!$ arrangements would give rise to $n_2!$ arrangements.

Therefore the number of arrangements would be $x \cdot n_1! \cdot n_2!$.

Similarly, if all the objects were unlike, the number of arrangements would be

$$x \cdot n_1! \cdot n_2! \cdots n_r!.$$

But if all n objects were unlike, the number of arrangements would be $n!$. Hence

$$x \cdot n_1! \cdot n_2! \cdots n_r! = n!.$$

$$x = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad (n \geq n_1 + n_2 + \cdots + n_r).$$

Therefore the number of n-arrangements of n objects, if n_1 are alike of one kind, n_2 alike of a second, $\cdots$, n_r alike of an rth, is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}.$$

Example 1. Find the number of ways of arranging the letters of the word Tennessee taken all at a time (a) if there is no restriction, (b) if the first two letters must be e.

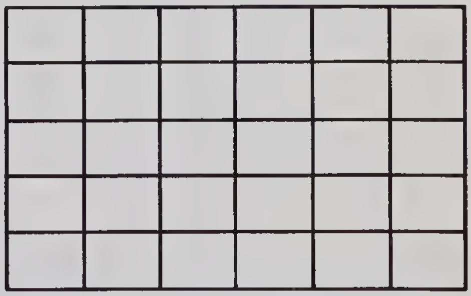
Solution:

(a)
$$\text{Number of arrangements} = \frac{9!}{4!2!2!} = 3780.$$

(b) Place two e's in the first two positions. Therefore

$$\text{number of arrangements} = 1 \times \frac{7!}{2!2!2!} = 630.$$

Example 2. A man wished to travel from one point in a city to a second point which is five blocks south and six blocks east of his starting point. In how many ways can he make the journey if he always travels either south or east?



Solution: If we consider a south-going route past one block as being represented by the letter S and an east-going route past one block as being represented by the letter E, then the problem is equivalent to the number of arrangements of eleven letters, five of which are S and six of which are E. Therefore

$$\text{number of routes} = \frac{11!}{6!5!} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} = 924.$$

Example 3. How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5, if the odd digits must always appear in ascending order?

Solution: Since the order of the odd digits cannot be altered, they may be considered as like digits. Hence

$$\begin{aligned}\text{number of numbers} &= \frac{5!}{3!}\\ &= 20\,.\end{aligned}$$

EXERCISE 2.3

1. Evaluate the following.

   (a) $\dfrac{6!}{3!2!}$ (b) $\dfrac{8!}{3!5!}$ (c) $\dfrac{10!}{6!2!}$ (d) $\dfrac{12!}{7!4!}$

2. Find the number of arrangements of all the letters of each of the following.

   (a) Ottawa (b) Toronto (c) subset
   (d) algebra (e) Mississippi (f) element

3. In how many ways can all the letters of the word *factor* be arranged if the consonants must always appear in the order *f*, *c*, *t*, *r*?

4. How many six-digit numbers may be formed from the digits 1, 1, 1, 2, 3, 4?

5. Find the number of arrangements of all the letters of the word *parallel*. In how many of the arrangements will the three *l*'s be together?

6. How many numbers greater than 3,000,000 can be formed from the digits 1, 2, 3, 3, 3, 4, 4?

7. How many numbers can be formed using all of the digits 1, 2, 2, 3, 3, 3, 4, if the odd digits must always occupy the odd positions?

8. How many possible routes may a person travel in order to go from one point in a city to a second point which is six blocks south and three blocks east, if he travels always in a southerly or easterly direction?

9. In the arrangements of all of the letters of the word *Toronto*, how many
   (a) start with the letter *o*?
   (b) start with two *o*'s?
   (c) start with one *o* with the second letter other than *o*?
   (d) start with three *o*'s?
   (e) start with two *o*'s with the third letter other than *o*?

10. In the arrangements of all of the letters of the word *Ottawa*, how many
    (a) start with *t*, with the second letter other than *t*?
    (b) start with *ta*?

11. How many numbers can be formed using all of the digits from 1 to 7 if the odd digits must always be in descending order and the even digits in ascending order?

12. How many odd numbers can be formed using all of the digits 1, 3, 4, 4, 5, 5?

2.4. Subsets of a Set

Consider the set $A = \{a, b, c, d, e\}$. One subset ϕ contains no elements; one subset, A itself, contains all five elements. It is also fairly obvious that there will be five different subsets each consisting of one element, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$. However, the number of subsets each consisting of 2 or 3 or 4 elements of A is not so obvious. Since $n(A) = 5$, it is possible, in this case, to write out all of the subsets consisting of, say, three elements of A. These subsets are $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{c, d, e\}$. However, even with just these ten possible subsets, it requires care and is time consuming to have to list all of them in order to find the required number of subsets. It would be rather tedious to find the number of 40-element subsets of a set of one hundred elements in this way!

Returning to our example of the number of 3-element subsets (usually simply called 3-subsets) of a given set of five elements. The subset $\{a, b, c\}$ gives rise to 3! arrangements of its elements all three at a time. This is true for each of the 3-subsets of A, and if we let the number of 3-subsets be x, the number of 3-arrangements of the five elements of A is $x(3!)$. However, we know that the total number of 3-arrangements of five elements is $5_{(3)}$. Hence

$$x(3!) = 5_{(3)}\,.$$

$$x = \frac{5_{(3)}}{3!}\,.$$

Therefore the number of 3-subsets is $\dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$. This agrees with the result of listing all the 3-subsets as we did above.

Extending this reasoning, we can develop a general method for finding the number of r-subsets of a set of n elements. We note of course, that $\{a, b, c\} = \{a, c, b\} = \{b, a, c\}$, etc. In selecting a subset of a given set, the order in which the elements are selected is immaterial. We can, therefore, define an r-subset as follows.

DEFINITION. An r-subset of a set of n elements is a selection of r of the elements *without regard to order* ($r \in W$, $n \in N$).

We could regard finding the number of r-arrangements of n objects as first finding the number of r-subsets of the set of n elements, and then finding the number of ways of rearranging the elements of each of the subsets.

To Find the Number of r-Subsets of a Set of n Elements

Let the number of r-subsets be x. Each of the x subsets can be rearranged in $r!$ different orders.

Therefore the number of r-arrangements is $x(r!)$.

But the number of r-arrangements of n objects is $n_{(r)}$. Hence

$$x(r!) = n_{(r)}.$$

$$x = \frac{n_{(r)}}{r!}.$$

Therefore the number of r-subsets is $\frac{n_{(r)}}{r!}$.

The symbol used for the number of r-subsets of a set of n elements is $\binom{n}{r}$. Therefore

$$\binom{n}{r} = \frac{n_{(r)}}{r!}.$$

Other symbols used are $C(n, r)$ and ${}_nC_r$. Also, since

$$n_{(r)} = \frac{n!}{(n-r)!},$$

therefore

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

Example 1. Find the number of 2-subsets of a set of twelve elements.

Solution:

$$\text{Number of 2-subsets} = \binom{12}{2} = \frac{12 \times 11}{2 \times 1} = 66$$

Example 2. Prove that $\binom{n}{r} = \binom{n}{n-r}$.

Solution:

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

$$\binom{n}{n-r} = \frac{n!}{[n-(n-r)]!\,(n-r)!} = \frac{n!}{(n-r)!\,r!}.$$

Therefore

$$\binom{n}{r} = \binom{n}{n-r}.$$

These examples indicate that in the more practical type of problem where an evaluation is necessary, as in Example 1, it is usually simpler to use the form $\binom{n}{r} = \frac{n_{(r)}}{r!}$, but in the more theoretical type of examples as in Example 2, the factorial form is usually the more useful.

Example 2 also gives a most useful result.

$$\binom{8}{6} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3}{6 \times 5 \times 4 \times 3 \times 2 \times 1}.$$

$$\binom{8}{2} = \frac{8 \times 7}{2 \times 1}.$$

From Example 2, we know that $\binom{8}{6} = \binom{8}{2}$ and thus to evaluate $\binom{8}{6}$, it is obviously simpler to evaluate $\binom{8}{2}$. This is no accident. For every subset of r elements selected from a set of n elements, there must be a corresponding set of the $(n - r)$ elements which were *not* selected. In the example of the 3-subsets of the set $A = \{a, b, c, d, e\}$ used at the beginning of this section, corresponding to the subset $\{a, b, c\}$ of the elements selected, there is the subset $\{d, e\}$ of the elements not selected. If we form the corresponding subsets of elements not selected for each subset selected, we will obtain the ten 2-subsets of the set A.

Example 3. Find the number of 18-subsets of a set of twenty elements.

Solution:

$$\begin{aligned}\text{Number of subsets} &= \binom{20}{18}\\ &= \binom{20}{2}\\ &= \frac{20 \times 19}{2 \times 1}\\ &= 190\,.\end{aligned}$$

Example 4. In a club with twelve members,

(a) in how many ways can a committee of three be selected?

(b) in how many ways can a president, secretary, and treasurer be appointed?

Solution:

(a) Number of ways of selecting committee

$$\begin{aligned}&= \binom{12}{3}\\ &= \frac{12 \times 11 \times 10}{3 \times 2 \times 1}\\ &= 220\,.\end{aligned}$$

(b) Number of ways of appointing officers

$$\begin{aligned} &= 12_{(3)} \\ &= 12 \times 11 \times 10 \\ &= 1320\,. \end{aligned}$$

We note that in the first part a committee of John, Jane, and Tom is exactly the same committee as Jane, Tom, and John. The order of selection is unimportant. This indicates that 3-subsets are required.

In the second part the appointment of John as president, Jane as secretary, and Tom as treasurer is *not* the same as Jane as president, Tom as secretary, and John as treasurer. The order is of importance and the number of 3-arrangements is required.

Example 5. In how many ways can six 3's and four 2's be arranged in a row so that the 2's are always apart?

Solution:

■ 3 ■ 3 ■ 3 ■ 3 ■ 3 ■ 3 ■

We can first arrange the 3's in a row. The 2's must now be placed in position. The seven possible positions for the 2's are marked ■ in the diagram and the four 2's may be placed in any four of these seven positions. Thus we require the number of 4-subsets of the seven positions. Therefore the number of ways of arranging the 3's and 2's is

$$\begin{aligned} \binom{7}{4} &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \\ &= 35\,. \end{aligned}$$

Example 6. Ten boys volunteered to assist in the organization of a school dance. It was agreed that three would be responsible for decorating, two for ticket sales, and the remaining five for clean-up after the dance. In how many ways could the ten boys be chosen for the different groups?

Solution: For the decorating group, 3 are selected from 10 in $\binom{10}{3}$ ways.

For tickets, 2 are selected from the remaining 7 in $\binom{7}{2}$ ways.

For clean-up, 5 are selected from the remaining 5 in $\binom{5}{5}$ ways.

Therefore the number of ways of selecting the groups is

$$\begin{aligned} \binom{10}{3}\binom{7}{2}\binom{5}{5} &= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \frac{7 \times 6}{2 \times 1} \times 1 \\ &= 2520\,. \end{aligned}$$

EXERCISE 2.4

1. Evaluate each of the following.

(a) $\binom{8}{6}$ (b) $\binom{5}{2}$ (c) $\binom{14}{12}$ (d) $\binom{9}{9}$ (e) $\binom{9}{0}$

(f) $\binom{21}{3}$ (g) $\binom{52}{50}$ (h) $\binom{100}{99}$ (i) $\binom{600}{1}$ (j) $\binom{15}{6}$

2. Find the number of 2-subsets of the set $\{0, 1, 2, 3\}$.

3. Find the number of 5-subsets of the set $\{a, b, c, d, e, f, g\}$.

4. For the set $\{1, 2, 3, 4\}$, find the number of 0-subsets, the number of 1-subsets, the number of 2-subsets, the number of 3-subsets, the number of 4-subsets. What is the total number of subsets of the given set?

5. (a) In how many ways can a committee of 3 be selected from 12 students?
   (b) In how many ways can a president, secretary, and treasurer be chosen from twelve students?

6. If fourteen points, no three of which are collinear, are marked on a sheet of paper, how many line segments can be drawn to join pairs of points?

7. How many diagonals has a polygon of twelve sides?

8. How many diagonals has a polygon of n sides?

9. Ten friends attend a reunion. Each shakes hands with each of the others. How many handshakes occur?

10. In how many ways can a committee of three men and two women be selected from eight men and six women?

11. In how many ways can a committee of five be selected from eight men and six women if at least three of the committee members must be men?

12. In how many ways can a committee of five be selected from ten men and seven women if at least one of the committee members must be a man?

13. The student council of Brandex High School consists of twenty members. In how many ways can a committee of four be selected if the president and secretary must be included?

14. In how many ways can a committee of four be selected from eight men and seven women if Miss Jones refuses to serve on the same committee as Mr. Smith?

15. In how many ways can twelve similar books be arranged on three shelves if each shelf must contain at least one book?

16. In how many ways can twelve similar books be arranged on three shelves if there is no restriction and one shelf may receive all of the books?

17. Twelve different books are to be shared so that Tom receives five, Dick receives four, and Harry receives three. In how many different ways can the books be distributed?

18. Solve for x: $\binom{x+1}{3} = \binom{x}{2}$.

19. Solve for x: $\binom{x}{2} = 28$.

20. Solve for x: $\binom{x}{3} = 3(x_{(2)})$.

21. Prove that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

22. Prove that $r\binom{n}{r} = n\binom{n-1}{r-1}$.

23. In how many ways can a bridge hand of thirteen cards be dealt from a deck of fifty-two cards? (Solution may be left in factorial form!)

24. In how many ways can thirteen cards be dealt to each of four players from a deck of fifty-two cards? (Solution may be left in factorial form!)

25. In how many ways can eight different books be divided into two parcels of five books in one parcel and three in the other?

26. In how many ways can eight different books be divided into parcels of four books in each parcel?

2.5. Number of Subsets of a Given Set

In question (4), Exercise 2.4, we were asked to find the number of 0-subsets, 1-subsets, 2-subsets, 3-subsets, and 4-subsets of the set $\{1, 2, 3, 4\}$. These are all the possible subsets of a set of four elements and the total number of subsets is

$$\begin{aligned}\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} &= 1 + 4 + 6 + 4 + 1 \\ &= 16\,.\end{aligned}$$

If the number of elements in the given set is large, it is obviously time consuming to calculate the total number of subsets in this way.

We can consider the problem in another way. Each element of the set $\{1, 2, 3, 4\}$ either is a member of a particular subset or is not a member. The first element may be either selected or not selected as a member of a subset. It can, therefore, be treated in two ways, selected or rejected. For each of these two ways, the second element may be treated in two ways, selected or rejected.

Therefore the first two elements may be treated in four ways (2 × 2). Similarly, the third element may be either selected or rejected in two ways for each of these four ways. Hence the first three elements may be treated in 2^3 ways, and the four elements, in 2^4 ways. The total number of subsets is, therefore, $2^4 = 16$. We note that this includes the possibility that none of the elements is selected (the empty set) or that all of the elements are selected (the given set itself). In particular problems one or other or both of these possibilities may have to be rejected.

By continuing the argument, we can show in the same way that the number of subsets of a set of n elements is 2^n.

The set consisting of all the subsets of a given set A is known as the *Power Set of A* and is symbolized by $\mathcal{P}(A)$.

Therefore, if

$$n(A) = n\,,$$

$$n[\mathcal{P}(A)] = 2^n\,.$$

If $A = \{1, 2, 3\}$,

$$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\,.$$

$$n(A) = 3$$

and

$$n[\mathcal{P}(A)] = 2^3 = 8\,.$$

Also we see that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n\,.$$

This last result will be proved in a different way in Chapter 4 (the Binomial Theorem).

Example 1. How many different sums of money can be obtained by selecting from a \$10 bill, a \$5 bill, a \$2 bill, and a \$1 bill?

Solution: Since the empty set is not a solution,

$$\begin{aligned}\text{the number of sums} &= 2^4 - 1\\ &= 16 - 1\\ &= 15\,.\end{aligned}$$

It may happen on certain occasions that we require the number of subsets of a set which contain repetitions of certain elements. Normally we do not repeat elements in a set, for instance $\{1, 2, 2, 2, 3\} = \{1, 2, 3\}$; but for some purposes we may wish to consider all five elements of this set and not simply the different elements.

Example 2. Find the number of different divisors of the number 2520, not including 1 or 2520.

Solution: If we first find all the prime factors of 2520, we see that

$$2520 = 2^3 \times 3^2 \times 5 \times 7\,.$$

The set of prime factors of 2520 is

$$\{2, 2, 2, 3, 3, 5, 7\}\,.$$

$$\begin{array}{r|r}
2 & 2520 \\ \hline
2 & 1260 \\ \hline
2 & 630 \\ \hline
3 & 315 \\ \hline
3 & 105 \\ \hline
5 & 35 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array}$$

The product of any number of these factors will be a divisor of 2520. For example, 2, 2×2, $2 \times 2 \times 3 \times 3 \times 5$, are divisors of 2520.

What we require, therefore, is the number of subsets of a set of seven elements of which three are alike of one kind (the three 2's) and two are alike of a second kind (the two 3's).

For any subset we may select no, one, two, or three 2's. Hence the 2's may be selected in four ways. Similarly for each of these ways, the two 3's may be selected in three ways.

Therefore the 2's and 3's may be selected in 4×3 ways, and for each of these ways, the remaining unlike numbers (5 and 7) may be selected in 2^2 ways.

Therefore, the number of subsets is $4 \times 3 \times 2^2$. However, this includes the empty set and also the set $\{2, 2, 2, 3, 3, 5, 7\}$ itself, which are not required. (The latter set gives 2520 as the product of its elements.)

$$\begin{aligned}
\text{Therefore the number of divisors} &= 4 \times 3 \times 2^2 - 2 \\
&= 48 - 2 \\
&= 46\,.
\end{aligned}$$

In the same way we may prove that the number of subsets of a set of n elements, of which p are alike of one kind, q are alike of a second kind, and r are alike of a third kind $(n \geq p + q + r)$, is

$$(p + 1)(q + 1)(r + 1)\,2^{n-p-q-r}\,.$$

The proof of this is asked in Exercise 2.5.

Example 3. How many selections, any number at a time, may be made from three red pencils, four blue pencils, one green pencil, and one white pencil if at least one must be chosen?

Solution:

$$\begin{aligned}
\text{Number of selections} &= (3 + 1)(4 + 1)\,2^2 - 1 \\
&= 4 \times 5 \times 4 - 1 \\
&= 79\,.
\end{aligned}$$

EXERCISE 2.5

1. State the number of subsets of each of the following sets.
   $A = \{a, b, c\}$ $\quad B = \{1, 2, 3, 4, 5\}$ $\quad C = \{x \in N \mid x < 10\}$
   $D = \{x \in I \mid 3 < x < 8\}$ $\quad E = \{x \in N \mid x < 30 \text{ and } x \text{ is prime}\}$
2. If $A = \{a, b, c, d\}$, list the elements of $\mathcal{P}(A)$.
3. If $n[\mathcal{P}(X)] = 128$, what is the value of $n(X)$?
4. How many different weights can be formed from weights of 1 lb., 2 lb., 4 lb., 8 lb., and 16 lb.?
5. How many different sums of money can be obtained by selecting from 1 penny, 1 nickel, 1 dime, and 1 quarter?
6. If $U = \{1, 2, 3, \cdots, 20\}$, $A = \{x \in U \mid x < 6\}$, $B = \{x \in U \mid 3 < x < 8\}$, $C = \{x \in U \mid x \text{ is prime}\}$, find the number of subsets of each of the following sets.
   (a) $A \cap B$ (b) $A \cup B$ (c) $C \cap B'$
   (d) $(B \cup C)'$ (e) $(A \cup C)'$ (f) $(A' \cup B') \cap C$
7. Find the number of different divisors of 432, not including 1 or 432.
8. Find the number of different divisors of 27,720, not including 1 or 27,720.
9. Find the number of subsets of $\{1, 1, 2, 2, 2, 3, 3, 4, 5\}$.
10. How many subsets of $\{1, 1, 2, 2, 2, 3, 3, 4, 5\}$ will contain all three 2's?
11. Find the number of subsets of {a, a, b, b, b, b, c}.
12. Prove that the number of subsets of a set of n elements, of which p are alike of one kind, q alike of a second kind, and r alike of a third kind, is
$$(p + 1)(q + 1)(r + 1)\, 2^{n-p-q-r} \quad (n \geq p + q + r).$$

2.6. Arrangements of Subsets

In many problems which require finding the number of arrangements (or permutations) of subsets of particular sets, it is necessary first of all to select the subsets before considering the possible arrangements of each group of subsets.

If we are asked to find the number of arrangements, each consisting of two vowels and three consonants, from the letters a, e, i, o, u, p, q, r, s, t, we are considering arrangements of letters selected from two sets, the set of vowels {a, e, i, o, u} and the set of consonants {p, q, r, s, t}.

We must first find the number of 2-subsets of the set of five vowels and then the number of 3-subsets of the set of five consonants. Each of those subsets may then be combined and the number of arrangements calculated.

Example 1. Find the number of 5-arrangements, each consisting of two vowels and three consonants, of the letters a, e, i, o, u, p, q, r, s, t.

Solution: The vowels may be selected in $\binom{5}{2}$ ways.

For each of these selections, the consonants may be selected in $\binom{5}{3}$ ways.

Therefore the number of selections is $\binom{5}{2}\binom{5}{3}$.

The letters in each selection may be arranged in 5! ways. Therefore

$$\begin{aligned}\text{number of 5-arrangements} &= \binom{5}{2}\binom{5}{3}\,5!\\ &= 10 \times 10 \times 120\\ &= 12{,}000\,.\end{aligned}$$

When elements of the given set are repeated, we know how to find the number of arrangements when *all* of the elements are used in each arrangement. When only some of the elements are used in each arrangement, we first have to select the possible subsets before we can find the number of arrangements.

Example 2. How many different four-digit numbers can be formed from the digits 1, 1, 1, 2, 2, 3, 3, 4, 5?

Solution:

Digits	Cases	Subsets	Arrangements
1, 1, 1 2, 2 3, 3 4 5	3 like, 1 different	$\binom{1}{1}\binom{4}{1} = 4$	$4 \times \frac{4!}{3!} = 16$
	2 like, 2 like	$\binom{3}{2} = 3$	$3 \times \frac{4!}{2!\,2!} = 18$
	2 like, 2 different	$\binom{3}{1}\binom{4}{2} = 18$	$18 \times \frac{4!}{2!} = 216$
	4 different	$\binom{5}{4} = 5$	$5 \times 4! = 120$

$$\begin{aligned}\text{Therefore the number of four-digit numbers} &= 16 + 18 + 216 + 120\\ &= 370\,.\end{aligned}$$

EXERCISE 2.6

1. How many arrangements, each consisting of three vowels and three consonants, can be made from five different vowels and ten different consonants?

2. From nine boys and six girls, four boys and three girls are to be selected to sit along one side of a table. How many seating arrangements are possible?
3. In question (2), how many seating arrangements are possible if the four boys and three girls are to sit around a circular table, if all the chairs are identical?
4. How many six-digit numbers, each consisting of three odd digits and three even digits, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9?
5. How many six-digit numbers, each consisting of three odd digits and three even digits, can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?
6. Find the number of 3-subsets of the set $\{a, a, a, b, b, c, d\}$.
7. How many arrangements, each consisting of three letters, can be made from the letters of the word *Toronto*?
8. How many different three-digit numbers can be formed from the digits 3, 3, 3, 4, 4, 5, 6, 7?
9. How many different four-digit numbers can be formed from five 3's, four 4's, two 5's, two 6's, one 7, and one 8?
10. How many different four-digit numbers, each greater than 3,000, can be formed from two 1's, three 2's, and two 3's?

Chapter Summary

Definitions

An *r-arrangement* of n objects is an ordered selection of r of the objects ($r \in W$, $n \in N$).

An *r-subset* of a set of n elements is a selection of r of the elements *without regard to order* ($r \in W$, $n \in N$).

Formulae and Symbols

(1)
$$n(A \times B) = n(A) \cdot n(B)\,.$$

(2) The number of r-arrangements of n objects is

$$\begin{aligned} n_{(r)} &= n(n-1)(n-2)\cdots(n-r+1) \qquad (n \geq r) \\ &= \frac{n!}{(n-r)!}, \end{aligned}$$

where $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ and $0! = 1$.

(3) The number of r-subsets of a set of n elements is

$$\binom{n}{r} = \frac{n_{(r)}}{r!} = \frac{n!}{r!(n-r)!}\,.$$

(4) The number of n-arrangements of n objects if n_1 are alike of one kind, n_2 alike of a second kind, $\cdots$, n_r alike of an rth kind is

$$\frac{n!}{n_1!n_2!\cdots n_r!} \qquad (n \geq n_1 + n_2 + \cdots + n_r)\,.$$

(5) $$\binom{n}{r} = \binom{n}{n-r}.$$

(6) The number of subsets of a set of n objects is 2^n, or, if $n(A) = n$, then $n[\mathcal{P}(A)] = 2^n$.

REVIEW EXERCISE 2

1. Evaluate each of the following.

(a) $7_{(3)}$ (b) $7_{(5)}$ (c) $100_{(2)}$ (d) $12_{(3)}$

(e) $\binom{7}{3}$ (f) $\binom{8}{8}$ (g) $\binom{100}{99}$ (h) $\binom{12}{5}$

(i) $\frac{12!}{10!}$ (j) $\frac{7 \times 5!}{3!}$ (k) $\frac{8!}{4!} - \frac{7!}{3!}$ (l) $\frac{8!}{3!} - \frac{7!}{4!}$

2. Simplify as far as possible.

(a) $\frac{x!}{(x-1)!}$ (b) $\frac{(x+1)!}{(x-1)!}$ (c) $\frac{(n-r)!}{(n-r-1)!}$

3. Prove that $\frac{(2n+2)!}{(n+1)!} = \frac{2(2n+1)!}{n!}$.

4. Prove that $5\binom{8}{5} = 8\binom{7}{4}$.

5. Prove that $\binom{n+1}{r+1} = \frac{n+1}{r+1}\binom{n}{r}$.

6. In how many ways can a committee of five be formed from six men and five women if
   (a) there is no restriction?
   (b) the committee must contain exactly three men?
   (c) the committee must contain at least one man?

7. A starting line-up for a hockey team must consist of one goalkeeper, two defencemen, and three forwards.
   (a) In how many ways can a starting line-up be selected from two goalkeepers, five defencemen, and nine forwards?
   (b) In how many ways can the different positions be filled if the same two goalkeepers, five defencemen, and nine forwards are available?

8. How many different arrangements can be made from all the letters of the word *distinct* if
   (a) there is no restriction?
   (b) the first and last letters must be alike?

9. How many different seven-digit numbers can be formed from the digits 1, 1, 1, 2, 2, 3, 3?

10. How many different seven-digit numbers can be formed from the digits 0, 0, 1, 1, 2, 2, 2?

11. How many different licence plates, each consisting of one letter and either four or five digits, can be formed from the letters H, J, K, L, M, and from the digits 0 to 9, if 0 cannot be the first digit and repetitions are allowed?

12. In how many ways can six different books be arranged in a row if two specified books must always be together?

13. In how many ways can twelve copies of the same book be arranged on three shelves if each shelf must contain at least one copy?

14. How many arrangements can be formed from all the letters of the word *combine* if the vowels must always be in the same order?

15. How many arrangements can be formed from all of the letters of the word *combine* if the vowels must occupy the second, fifth, and seventh positions?

16. An examination consists of three compulsory questions and another six questions of which the candidate may select any four. In how many ways may the candidate select the questions to be answered?

17. Find the number of divisors of 6006, excluding 1 and 6006.

18. Find the number of divisors of 3150, excluding 1 and 3150.

19. In how many ways can a committee of four be selected from twelve members?

20. In how many ways can a committee, consisting of a president, a secretary, and two other members, be chosen from twelve members?

21. In how many ways can twelve people be divided into three groups consisting of four, five, and three individuals?

22. In how many ways can twelve people be divided into three groups, each consisting of four individuals?

23. How many arrangements, each consisting of two vowels and two consonants, can be formed from the letters a, e, i, o, u, b, c, d, f?

24. How many diagonals has an octagon?

25. How many sides has a polygon if it has thirty-five diagonals?

26. Solve the following equations.

    (a) $\binom{n}{2} = 45$ (b) $n_{(3)} = 2\binom{n}{2}$

    (c) $n! = 5(n-1)!$ (d) $10\binom{n}{r-1} = 3\binom{n}{r};\ 4\binom{n}{r+1} = 9\binom{n}{r}$

27. How many different three-digit numbers can be formed from three 4's, three 3's, two 2's, and two 1's?

28. How many arrangements of four letters each can be formed from the letters of the word *Tennessee*?

29. How many different four-digit numbers, each greater than 5,000, can be formed from three 2's, two 3's, two 4's, and two 5's?

30. How many different three-digit numbers, each greater than 400, can be formed from three 1's, two 2's, two 4's, and one 5?

Chapter 3

MATHEMATICAL INDUCTION

3.1. Sigma Notation

If we consider the finite series

$$1 + 3 + 5 + 7 + 9 + 11\,,$$

we see that it can be rewritten as

$$[2(1) - 1] + [2(2) - 1] + [2(3) - 1] + [2(4) - 1] + [2(5) - 1] + [2(6) - 1]\,.$$

Each term of the series is of the form $2k - 1$ and the terms of the series can be formed by letting k take, in turn, the value of each member of the set

$$\{1, 2, 3, 4, 5, 6\}\,.$$

A very useful shorthand notation to indicate such a series is obtained by using the Greek capital letter $\sum$ (sigma), called the summation sign. In our example we can write

$$1 + 3 + 5 + 7 + 9 + 11 = \sum_{k=1}^{6}(2k - 1).$$

This is read as "the sum of $2k - 1$ from $k = 1$ to $k = 6$."

1 and 6 are called the *lower* and *upper* limits of summation; $\{1, 2, 3, 4, 5, 6\}$ is called the *domain of summation*. The *summand* is $2k - 1$.

In the same way,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

In this case the lower and upper limits of summation are 1 and n.

The summation symbol $\sum$ may sometimes be used to indicate the sum of the series as well as the series itself. In order to make this clear we shall use expressions such as "compute", "evaluate", or "find the sum of" when the sum of the series is required.

Example 1. Express $\sum_{k=1}^{6}(k+1)$ in expanded form.

Solution:

$$\begin{aligned}\sum_{k=1}^{6}(k+1) &= (1+1)+(2+1)+(3+1)+(4+1)+(5+1)+(6+1)\\ &= 2+3+4+5+6+7\,.\end{aligned}$$

Example 2. Evaluate $\sum_{n=1}^{5} 2^n$.

Solution:

$$\begin{aligned}\sum_{n=1}^{5} 2^n &= 2^1+2^2+2^3+2^4+2^5\\ &= 2+4+8+16+32\\ &= 62\,.\end{aligned}$$

Example 3. Prove that $\sum_{k=1}^{n} 3k^2 = 3\sum_{k=1}^{n} k^2$.

Solution:

$$\begin{aligned}\sum_{k=1}^{n} 3k^2 &= 3(1)^2+3(2)^2+3(3)^2+\cdots+3n^2\\ &= 3(1^2+2^2+3^2+\cdots+n^2)\\ &= 3\sum_{k=1}^{n} k^2\,.\end{aligned}$$

EXERCISE 3.1

1. Express each of the following series in expanded form.

(a) $\sum_{i=1}^{6} i$ (b) $\sum_{j=1}^{6} j$ (c) $\sum_{n=1}^{7}(2-n)$

(d) $\sum_{n=1}^{8}(2n+3)$ (e) $\sum_{k=1}^{4} 3k$ (f) $\sum_{n=1}^{7}|2-n|$

(g) $\sum_{i=1}^{6}(-1)^i$ (h) $\sum_{k=1}^{n} k^2$ (i) $\sum_{k=1}^{n}\frac{k(k+1)}{2}$

2. By expanding each series, prove the following results.

(a) $\sum_{k=3}^{8} k = \sum_{k=1}^{6}(k+2)$

(b) $\sum_{i=3}^{7} i^2 = \sum_{i=1}^{5}(i+2)^2$

(c) $\sum_{j=0}^{5}(j+2) = \sum_{j=1}^{6}(j+1)$

(d) $\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}$

(e) $\sum_{n=1}^{6}(3n^2+n) = 3\sum_{n=1}^{6} n^2 + \sum_{n=1}^{6} n$

(f) $\sum_{n=1}^{8}(n+1) = 8 + \sum_{n=1}^{8} n$

3. Evaluate each of the following.

(a) $\sum_{k=1}^{8} k$ (b) $\sum_{t=1}^{5} t^2$ (c) $\sum_{n=2}^{6} 2^{n-2}$

(d) $\sum_{n=0}^{7}(2n+1)$ (e) $\sum_{i=1}^{10}(-1)^i$ (f) $\sum_{i=1}^{11}(-1)^i$

4. Write each of the following series using sigma notation.

(a) $1+2+3+4+5+6$
(b) $1+4+9+16+25+36+49$
(c) $2+4+8+16+32+64$
(d) $2-4+8-16+32-64$
(e) $a_1^2+a_2^2+a_3^2+\cdots+a_n^2$
(f) $ab+a^2b^2+a^3b^3+a^4b^4$
(g) $a+a^2b+a^3b^2+a^4b^3+a^5b^4$
(h) $3+7+11+15+19+23$
(i) $2+6+18+54+162$

3.2. Properties of Summation

In the previous section, some of the examples should have suggested the possibility of certain properties which could be used to simplify operations using sigma notation. In this section we shall deduce these properties and formally prove them.

The expression $\sum_{i=1}^{n} 3$ states that we have to write a series of n terms, each term being 3. Therefore

$$\begin{aligned}\sum_{i=1}^{n} 3 &= 3+3+3+\cdots \text{ to } n \text{ terms}\\ &= 3n\,.\end{aligned}$$

This will be obviously true for any constant and we obtain

Summation Property 1.

$$\sum_{i=1}^{n} a = a + a + a + \cdots \text{ to } n \text{ terms}$$
$$= na .$$

One fairly trivial property, but one which is frequently forgotten, is

Summation Property 2.

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j .$$

This is very easily proved.

Proof:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n .$$

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + a_3 + \cdots + a_n .$$

Therefore

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j .$$

In Section 3.1 we proved that

$$\sum_{k=1}^{n} 3k^2 = 3 \sum_{k=1}^{n} k^2 .$$

This can now be extended to the more general form of

Summation Property 3.

$$\sum_{i=1}^{n} a(b_i) = a \sum_{i=1}^{n} b_i .$$

Proof:

$$\sum_{i=1}^{n} a(b_i) = ab_1 + ab_2 + ab_3 + \cdots + ab_n$$
$$= a(b_1 + b_2 + b_3 + \cdots + b_n) .$$

Therefore

$$\sum_{i=1}^{n} a(b_i) = a \sum_{i=1}^{n} b_i .$$

Question (2e), Exercise 3.1 asked us to prove that

$$\sum_{n=1}^{6} (3n^2 + n) = 3 \sum_{n=1}^{6} n^2 + \sum_{n=1}^{6} n .$$

This again is a particular case of a combination of Property 3 and

Summation Property 4.

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\,.$$

Proof:

$$\sum_{i=1}^{n}(a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$$

$$= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)\,.$$

Therefore

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\,.$$

A final and more subtle property which is often very useful permits us to add the same constant (positive or negative) to the lower *and* upper limits of summation.

Summation Property 5.

$$\sum_{i=m}^{n} a_i = \sum_{i=m+p}^{n+p} a_{i-p}\,.$$

Proof:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n\,.$$

$$\sum_{i=m+p}^{n+p} a_{i-p} = a_{m+p-p} + a_{m+p-p+1} + \cdots + a_{n+p-p}$$

$$= a_m + a_{m+1} + \cdots + a_n\,.$$

Therefore

$$\sum_{i=m}^{n} a_i = \sum_{i=m+p}^{n+p} a_{i-p}\,.$$

This property is probably most frequently used to change the lower limit of summation to 1. For example,

$$\sum_{k=6}^{20} k^3 = \sum_{k=1}^{15} (k+5)^3\,.$$

-5 is added to the lower limit to produce 1. Hence -5 must also be added to the upper limit and $+5$ to the summation variable in the summand.

Another method to produce the same result if we wish to evaluate the series is

$$\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i\,.$$

The proof of this is required in Exercise 3.2 but an example should demonstrate its plausibility.

$$\sum_{k=6}^{20} k = \sum_{k=1}^{20} k - \sum_{k=1}^{5} k\,.$$

This states that the sum of all the positive integers from 6 to 20 is equal to the sum of all the positive integers from 1 to 20 less the sum of the positive integers from 1 to 5.

In the examples we will quote by number the property or properties being used. This is only for the purpose of emphasizing the authority until we become familiar with it, and after some practice, it may be omitted. In the same way certain steps in the development may easily be omitted after some facility has been acquired.

Example 1. Express $\sum_{k=1}^{n} (3k - 1)^2$ in terms of monomial summations.

Solution:

$$\begin{aligned} \sum_{k=1}^{n} (3k-1)^2 &= \sum_{k=1}^{n} (9k^2 - 6k + 1) \\ &= \sum_{k=1}^{n} 9k^2 - \sum_{k=1}^{n} 6k + \sum_{k=1}^{n} 1 && \text{property 4} \\ &= 9\sum_{k=1}^{n} k^2 - 6\sum_{k=1}^{n} k + n\,. && \text{properties 3 and 1} \end{aligned}$$

Example 2. Prove that $\sum_{k=6}^{12} k^2 = \sum_{k=1}^{7} k^2 + 10\sum_{k=1}^{7} k + 175\,.$

Solution:

$$\begin{aligned} \sum_{k=6}^{12} k^2 &= \sum_{k=1}^{7} (k+5)^2 && \text{property 5} \\ &= \sum_{k=1}^{7} (k^2 + 10k + 25) \\ &= \sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} 10k + \sum_{k=1}^{7} 25 && \text{property 4} \\ &= \sum_{k=1}^{7} k^2 + 10\sum_{k=1}^{7} k + 175\,. && \text{properties 3 and 1} \end{aligned}$$

EXERCISE 3.2

1. Express the following in terms of monomial summations.

(a) $\sum_{i=1}^{n} (3a_i + b_i)$ (b) $\sum_{k=1}^{10} (2k^2 - 6k)$ (c) $\sum_{k=1}^{n} (k^3 + 6k^2 - 5)$

(d) $\sum_{p=1}^{n} (2^p + 3p)$ (e) $\sum_{i=1}^{10} (-1)^i(i^2 + 2i)$ (f) $\sum_{k=1}^{n} (2k^3 - 6k^2 + k - 2)$

2. Simplify each of the following.

(a) $\sum_{i=1}^{n} (3i + 2) + \sum_{k=1}^{n} (3k + 2)$

(b) $\sum_{a=7}^{12} a^2 - \sum_{b=1}^{6} (b^2 + 12b + 36)$

3. Rewrite each of the following with 1 as the lower limit of summation.

(a) $\sum_{k=12}^{20} k$ (b) $\sum_{a=-5}^{5} \frac{a}{a+6}$ (c) $\sum_{n=3}^{8} \frac{n+1}{n-2}$

(d) $\sum_{p=0}^{9} (p + 1)$ (e) $\sum_{k=7}^{10} 2k^2$ (f) $\sum_{i=-2}^{3} (i^2 + i)$

4. Prove that $\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i$.

3.3. Induction

If we consider the arithmetic series

$$1 + 3 + 5 + \cdots + (2n - 1) + \cdots,$$

we see that

$$\begin{aligned} S_1 &= 1 &&= 1 = 1^2, \\ S_2 &= 1 + 3 &&= 4 = 2^2, \\ S_3 &= 1 + 3 + 5 &&= 9 = 3^2, \\ S_4 &= 1 + 3 + 5 + 7 &&= 16 = 4^2. \end{aligned}$$

From the results of these four sums, it would appear that

$$S_n = 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for all } n \in N.$$

In coming to this conclusion from a few cases, we are using the method of induction. We cannot be certain that the result will hold for all $n \in N$. The more cases we consider, the more certain we become that our result is probably correct, but we can never test it for every positive integer. In spite of this lack of absolute certainty, induction is a very valuable mathematical and scientific tool. It can lead us to probable results which may be worth the effort of trying to prove deductively. We must, however, keep in mind that a result obtained inductively can never be finally accepted until it has been proved deductively. The most that we can say for the conclusion given above is that it appears *likely* that

$$\sum_{k=1}^{n} (2k - 1) = n^2.$$

Many problems in mathematics may first be visualized by examining a number of special cases and seeing a general pattern emerge. From this a general result may be conjectured. A deductive proof of this result may or may not be possible.

Goldbach's conjecture states that every even integer greater than 4 may be written as the sum of two odd primes in at least one way. So far no exception has been found, but neither has a deductive proof of the statement.

$$6 = 3 + 3, \qquad 8 = 5 + 3, \qquad 10 = 7 + 3 = 5 + 5,$$
$$12 = 7 + 5, \qquad 26 = 13 + 13 = 19 + 7 = 23 + 3,$$

are all examples which support the conjecture. In this case, the proof of the conjecture has eluded mathematicians, but in many cases the main difficulty arises in formulating the conjecture.

It is impossible to lay down hard and fast rules for reaching an inductive conclusion. A sense of pattern has to be developed and, in the case of sequences and series, it is often valuable to compare the various results for particular cases with the sequence of positive integers.

Example 1. Attempt to derive a general formula to evaluate

$$\sum_{k=1}^{n} \frac{k(k+1)}{2}.$$

Solution:

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1 \times 2}{2} + \frac{2 \times 3}{2} + \frac{3 \times 4}{2} + \frac{4 \times 5}{2} + \frac{5 \times 6}{2} + \cdots$$
$$= 1 + 3 + 6 + 10 + 15 + \cdots.$$

We see that

$$S_1 = 1,$$
$$S_2 = 4,$$
$$S_3 = 10,$$
$$S_4 = 20,$$
$$S_5 = 35.$$

Probably no pattern is immediately obvious but we might notice that $S_4 = 20 = 4 \times 5$. This might lead us to begin by writing $n(n+1)$ for S_n, even though it is obviously not true for other values. However, it is a start. Let us set these under each partial sum.

S_1	S_2	S_3	S_4	S_5
1	4	10	20	35
1×2	2×3	3×4	4×5	5×6

We might now look at S_5 and note that $35 = 5 \times 7$. This can be obtained by multiplying 5×6 by 7 and dividing the result by 6. This is also successful for S_4; so let us try it for each case.

S_1	S_2	S_3	S_4	S_5
1	4	10	20	35
$\frac{1 \times 2 \times 3}{6}$	$\frac{2 \times 3 \times 4}{6}$	$\frac{3 \times 4 \times 5}{6}$	$\frac{4 \times 5 \times 6}{6}$	$\frac{5 \times 6 \times 7}{6}$

We see that the result holds in each case and that it appears to be true that

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

We now test our conclusion for $n = 6$ and $n = 7$. If it is still true, we may feel more confident that our general formula is correct. However, we must remain conscious of the fact that it is still only a conjecture. Our next problem will be to attempt a proof that it is indeed true for all $n \in N$.

EXERCISE 3.3

By inductive methods, attempt to derive a general formula to evaluate each of the following.

1. $1 + 2 + 3 + \cdots + n$
2. $\sum_{k=1}^{n} \frac{1}{k(k+1)}$
3. $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)}$
4. $\sum_{i=1}^{n} \frac{2}{(2i-1)(2i+1)}$
5. $\sum_{k=1}^{n} \frac{1}{(3k-1)(3k+2)}$
6. $\sum_{p=1}^{n} (3p+1)$
7. $\sum_{k=1}^{n} \frac{1}{(4k-2)(4k+2)}$

Assuming that the following sequences continue in the most obvious manner, formulate an expression for the nth term (t_n).

8. $1, 4, 7, 10, \cdots$
9. $1, 2, 4, 8, \cdots$
10. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$
11. $3, 7, 15, 31, \cdots$
12. $\frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \frac{24}{5}, \cdots$
13. $\frac{7}{3}, \frac{19}{9}, \frac{55}{27}, \frac{163}{81}, \cdots$
14. $1, 3, 6, 10, \cdots$
15. $1, 5, 19, 65, \cdots$
16. For a certain sequence, $t_n = n^3 - 6n^2 + 14n - 8$. Evaluate t_1, t_2, t_3 and attempt to derive a simpler formula for t_n which will give the same first three terms but a different fourth term.

3.4. Proof by Mathematical Induction

In the previous section we predicted that

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2 .$$

We came to this conclusion by examining a limited number of special cases, but we could not be certain that the proposition would hold for all $n \in N$. We *do know,* however, that the proposition is true for a set S of some natural numbers where $S \subseteq N$. Certainly it is true for $n = 1$ and the problem that remains is to prove that $S = N$. To do this we make use of the *Axiom of Mathematical Induction.*

A set S of natural numbers is the set N of all natural numbers if

1. $1 \in S$,
2. $k + 1 \in S$ whenever $k \in S$.

We note that, if $k + 1 \in S$ whenever $k \in S$, then, since $1 \in S$, $1 + 1 = 2 \in S$. Since $2 \in S$, $2 + 1 = 3 \in S$, and so on. S is the set of all natural numbers and $S = N$.

To use the axiom to determine whether a set S of natural numbers is, in fact, the set N of all natural numbers, we must prove *two* things about S:

1. that $1 \in S$,
2. that, if we assume that $k \in S$ (the induction hypothesis), then it follows that $k + 1 \in S$.

If *both* of these can be established, then $S = N$.

In the example, to establish that $\sum_{i=1}^{n} (2i - 1) = n^2$ is true for all $n \in N$, we must first establish that the proposition is true for $n = 1$. When this is established, then

$$1 \in S.$$

Secondly, if we assume that the result is true for $k \in S$, (induction hypothesis) where $S \subseteq N$, that is,

$$\sum_{i=1}^{k} (2i - 1) = k^2 ,$$

then we must prove that $k + 1$ is also an element of S (that is, $k + 1 \in$ S). We must establish that

$$\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2 .$$

We can form the sum of $k+1$ terms by adding the $(k+1)$st term to the sum of k terms. From our assumption that

$$\sum_{i=1}^{k}(2i-1) = k^2,$$

we can state that

$$\sum_{i=1}^{k}(2i-1) + t_{k+1} = k^2 + t_{k+1};$$

that is,

$$\sum_{i=1}^{k+1}(2i-1) = k^2 + t_{k+1}.$$

What we now have to do is prove that the right side is indeed $(k+1)^2$.

Example 1. Prove that $\sum_{i=1}^{n}(2i-1) = n^2$.

Solution: Let S be the set of natural numbers for which $\sum_{i=1}^{n}(2i-1) = n^2$.

1. $$\sum_{i=1}^{1}(2i-1) = 2(1) - 1 = 1 = 1^2.$$

Therefore

$$1 \in S.$$

2. Assume that $k \in S$.

$$\sum_{i=1}^{k}(2i-1) = k^2.$$

Then

$$\sum_{i=1}^{k}(2i-1) + t_{k+1} = k^2 + t_{k+1}.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= k^2 + 2(k+1) - 1\\ &= k^2 + 2k + 1\\ &= (k+1)^2.\end{aligned}$$

Therefore

$$k+1 \in S \text{ if } k \in S \text{ and } 1 \in S.$$

Therefore

$$S = N,$$

and so

$$\sum_{i=1}^{n}(2i-1) = n^2 \quad \text{for all } n \in N.$$

Example 2. Prove that the sum of n terms of the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

is equal to

$$\frac{a(r^n - 1)}{r - 1}, \quad (r \neq 1).$$

Solution: Let T be the set of natural numbers for which $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

1. $$S_1 = \frac{a(r - 1)}{r - 1} = a \qquad (r \neq 1).$$

Therefore

$$1 \in T.$$

2. Assume that $k \in T$.

$$S_k = \frac{a(r^k - 1)}{r - 1}.$$

Therefore

$$S_k + t_{k+1} = \frac{a(r^k - 1)}{r - 1} + ar^k$$

and

$$S_{k+1} = \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

$$= \frac{a(r^k - 1 + r^{k+1} - r^k)}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}.$$

Therefore

$$k + 1 \in T \text{ if } k \in T \text{ and } 1 \in T.$$

Therefore

$$T = N,$$

and so

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for all } n \in N.$$

Before considering our next example we should consider some properties of the integers. If a and b are two integers, both divisible by an integer n, then $a + b$, $a - b$, and ab are also divisible by n. These results are easily proved.

Since a and b are divisible by n, let $a = nx$ and $b = ny$. Then

$$a + b = nx + ny = n(x + y), \text{ which is divisible by } n;$$

$$a - b = nx - ny = n(x - y), \text{ which is divisible by } n;$$

$$ab = nx \cdot ny = n^2xy, \text{ which is divisible by } n.$$

Note that $a/b = nx/ny = x/y$ is not necessarily divisible by n.

Example 3. Prove that $3^{4n} - 1$ is divisible by 80 for all $n \in N$.

Solution: Let T be the set of all natural numbers for which $3^{4n} - 1$ is divisible by 80.

1. $$3^{4(1)} - 1 = 81 - 1 = 80.$$

Therefore

$$1 \in T.$$

2. Assume that $k \in T$. Therefore

$$3^{4k} - 1 \text{ is divisible by } 80.$$

To show that $3^{4(k+1)} - 1$ is also divisible by 80, we consider the difference between $3^{4(k+1)} - 1$ and $3^{4k} - 1$.

$$\begin{aligned}(3^{4(k+1)} - 1) - (3^{4k} - 1) &= 3^{4k+4} - 1 - 3^{4k} + 1 \\ &= 3^{4k+4} - 3^{4k} \\ &= 3^{4k}(3^4 - 1) \\ &= 3^{4k}(80).\end{aligned}$$

Therefore if $3^{4k} - 1$ is divisible by 80, $3^{4(k+1)} - 1$ is also divisible by 80. Hence

$$k + 1 \in T \text{ if } k \in T \text{ and } 1 \in T.$$

Therefore

$$T = N,$$

and so

$$3^{4n} - 1 \text{ is divisible by } 80 \text{ for all } n \in N.$$

We should note that, although this method of proof is given the name mathematical induction, it is a form of *deductive* reasoning and that both parts of the proof are essential. For example, if we are asked to prove that

$$\sum_{i=1}^{n} (2i - 1) = 3n^2 - 6n + 4,$$

we see that

$$\sum_{i=1}^{1} (2i - 1) = 1,$$

and

$$3(1)^2 - 6(1) + 4 = 1.$$

$$\sum_{i=1}^{2} (2i - 1) = 1 + 3 = 4,$$

and

$$3(2)^2 - 6(2) + 4 = 4.$$

Hence the proposition is true for $n = 1$ and $n = 2$.

It is *not* true, however, for $n = 3$, and this becomes apparent when we attempt part 2 of the proof. We are unable to prove that if

$$\sum_{i=1}^{k} (2i - 1) = 3k^2 - 6k + 4\,,$$

then

$$\sum_{i=1}^{k+1} (2i - 1) = 3(k + 1)^2 - 6(k + 1) + 4\,.$$

More formally, let S be the set of all natural numbers for which

$$\sum_{i=1}^{n} (2i - 1) = 3n^2 - 6n + 4\,.$$

1.

$$2 - 1 = 3 - 6 + 4.$$

Therefore

$$1 \in S\,.$$

2. Assume $k \in S$.

$$\sum_{i=1}^{k} (2i - 1) = 3k^2 - 6k + 4\,.$$

Therefore

$$\sum_{i=1}^{k} (2i - 1) + t_{k+1} = 3k^2 - 6k + 4 + 2(k + 1) - 1\,.$$

$$\begin{aligned}\sum_{i=1}^{k+1} (2i - 1) &= 3k^2 - 4k + 5 \\ &= 3(k^2 + 2k + 1) - 10k + 2 \\ &= 3(k + 1)^2 - 6(k + 1) - 4k + 8 \\ &= 3(k + 1)^2 - 6(k + 1) + 4 \quad \textit{only} \text{ for } k = 1\,.\end{aligned}$$

Therefore

$$S \neq N\,,$$

and so

$$\sum_{i=1}^{n} (2i - 1) \neq 3n^2 - 6n + 4 \text{ for all } n \in N\,.$$

Similarly, if part 1 of the proof fails, there is no point in attempting part 2. To prove only part 2 may lead to erroneous conclusions. For example, if we are asked to prove

$$\sum_{i=1}^{n} (2i - 1) = n^2 + 5\,.$$

If S is the set of all natural numbers for which $\sum_{i=1}^{n} (2i - 1) = n^2 + 5$, obviously $1 \notin S$. We need go no farther.

Using only part 2, however, assume $k \in S$. Then

$$\sum_{i=1}^{k} (2i-1) = k^2 + 5,$$

and therefore

$$\sum_{i=1}^{k} (2i-1) + t_{k+1} = k^2 + 5 + 2(k+1) - 1.$$

$$\begin{aligned}\sum_{i=1}^{k+1} (2i-1) &= k^2 + 2k + 6 \\ &= (k+1)^2 + 5.\end{aligned}$$

Therefore

$$k + 1 \in S \text{ if } k \in S.$$

This would lead to a false conclusion since the proposition is false for all $n \in N$.

EXERCISE 3.4

Prove each of the following by mathematical induction.

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$.
3. $a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$.
4. $\sum_{p=1}^{n} (5p-1) = \frac{n(5n+3)}{2}$.
5. $\sum_{i=1}^{n} (3i-2) = \frac{n(3n-1)}{2}$.
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
7. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
8. $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.
9. $\sum_{p=1}^{n} \frac{1}{(3p-2)(3p+1)} = \frac{n}{3n+1}$.
10. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

11. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$.

12. $\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$.

13. Prove that $n(n + 1)(n + 2)$ is divisible by 6 for all $n \in N$.

14. Prove that $5^{2n} - 1$ is divisible by 3 for all $n \in N$.

Show that each of the statements (15-18) is false.

15. $\sum_{i=1}^{n} 2i = 2(n^2 - n + 1)$.

16. $\sum_{i=1}^{n} 2i = n^2 + n - 1$.

17. $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots + \frac{1}{n(n+1)} = \frac{2n+1}{n+1}$.

18. $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots + \frac{1}{n(n+1)} = \frac{n^3 - 2n^2 - 2}{n+1}$.

19. For the sequence defined by $t_1 = 1, t_{n+1} = 1 + \sqrt{1 + t_n}$, prove that
    (a) $t_n < 3$,
    (b) $t_{n+1} > t_n$ for all $n \in N$.

20. For the sequence defined by $t_1 = 4$, $t_{n+1} = \sqrt{t_n + 2}$, prove that
    (a) $t_n > 2$,
    (b) $t_{n+1} < t_n$ for all $n \in N$.

21. For the sequence defined by $t_1 = 2$, $t_{n+1} = \sqrt{2t_n + 5}$, prove that
    (a) $t_n < 4$
    (b) $t_{n+1} > t_n$ for all $n \in N$.

3.5. Sum of a Series

In Exercise 3.4 we were asked to prove certain basic summation formulae by mathematical induction. Among the results proved were the following.

(1) The sum of the arithmetic series

$$a + (a + d) + (a + 2d) + \cdots + a + (n - 1)d = \frac{n}{2}[2a + (n - 1)d].$$

(2) The sum of the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \qquad (r \neq 1).$$

(3) The sums of the natural numbers, of the squares of the natural numbers, and of the cubes of the natural numbers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

These formulae can be used to evaluate the sums of other series and may all be proved by methods other than mathematical induction. The first two have been proved in earlier grades but we will repeat the proof of (2).

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

$$r\sum_{k=1}^{n} ar^{k-1} = \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

$$\sum_{k=1}^{n} ar^{k-1} - r\sum_{k=1}^{n} ar^{k-1} = a - ar^n.$$

Therefore

$$(1-r)\sum_{k=1}^{n} ar^{k-1} = a(1-r^n)$$

and

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}, \qquad (r \neq 1).$$

The sum of the natural numbers can be evaluated from the formula for the sum of the arithmetic series since the natural numbers form an arithmetic series with $a = 1,\ d = 1$.

$$\begin{aligned}\sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + n\\ &= \frac{n}{2}[2(1) + (n-1)(1)]\\ &= \frac{n}{2}(n+1)\\ &= \frac{n(n+1)}{2}.\end{aligned}$$

For the second formulae in (3), we may use the following algebraic identity.

$$\begin{aligned} n^3 - (n-1)^3 &= n^3 - (n^3 - 3n^2 + 3n - 1) \\ &= 3n^2 - 3n + 1 . \end{aligned}$$

$$\begin{aligned} 1^3 - 0^3 &= 3(1)^2 - 3(1) + 1 , \\ 2^3 - 1^3 &= 3(2)^2 - 3(2) + 1 , \\ 3^3 - 2^3 &= 3(3)^2 - 3(3) + 1 , \\ &\cdots\cdots\cdots\cdots\cdots \\ (n-1)^3 - (n-2)^3 &= 3(n-1)^2 - 3(n-1) + 1 , \\ n^3 - (n-1)^3 &= 3(n^2) - 3(n) + 1 . \end{aligned}$$

Adding, we obtain

$$\begin{aligned} n^3 &= 3(1^2 + 2^2 + \cdots + n^2) - 3(1 + 2 + \cdots + n) + n \\ &= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + n . \end{aligned}$$

Therefore

$$\begin{aligned} 3\sum_{k=1}^{n} k^2 &= n^3 + 3\sum_{k=1}^{n} k - n \\ &= n^3 + \frac{3n(n+1)}{2} - n \\ &= \frac{2n^3 + 3n(n+1) - 2n}{2} \\ &= \frac{n(2n^2 + 3n + 3 - 2)}{2} \\ &= \frac{n(2n^2 + 3n + 1)}{2} . \end{aligned}$$

$$3\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{2} .$$

Therefore

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} .$$

The method has an obvious advantage over the method of mathematical induction in that it yields the formula directly without the necessity of first having to obtain it by inductive reasoning or "educated guessing". However, the identity $n^3 - (n-1)^3 = 3n^2 - 3n + 1$ does appear apparently "out of the blue", and there is no obvious way of discovering its use other than by an inspired hunch.

Using these formulae and the rules of summation established in Section 3.2, we can evaluate the sums of many other series.

Example 1. Evaluate $\sum_{k=1}^{n}(4k^2 - 3k + 2)$.

Solution:

$$\begin{aligned}
\sum_{k=1}^{n}(4k^2 - 3k + 2) &= 4\sum_{k=1}^{n}k^2 - 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}2 \\
&= \frac{4n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + 2n \\
&= \frac{4n(n+1)(2n+1) - 9n(n+1) + 12n}{6} \\
&= \frac{n(8n^2 + 12n + 4 - 9n - 9 + 12)}{6} \\
&= \frac{n(8n^2 + 3n + 7)}{6}.
\end{aligned}$$

Example 2. Evaluate $\sum_{k=1}^{n}[2(3^k) - 1]$.

Solution:

$$\begin{aligned}
\sum_{k=1}^{n}[2(3^k) - 1] &= 2\sum_{k=1}^{n}3^k - \sum_{k=1}^{n}1 \\
&= \frac{2(3^{n+1} - 1)}{3 - 1} - n \\
&= 3^{n+1} - (n+1).
\end{aligned}$$

Example 3. Evaluate $\sum_{k=10}^{20}(2k^2 - k)$.

Solution:

$$\begin{aligned}
\sum_{k=1}^{n}(2k^2 - k) &= 2\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k \\
&= \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \\
&= \frac{2n(n+1)(2n+1) - 3n(n+1)}{6} \\
&= \frac{n(n+1)(4n+2-3)}{6} \\
&= \frac{n(n+1)(4n-1)}{6}.
\end{aligned}$$

$$\begin{aligned}
\sum_{k=10}^{20}(2k^2 - k) &= \sum_{k=1}^{20}(2k^2 - k) - \sum_{k=1}^{9}(2k^2 - k) \\
&= \frac{20(21)(79)}{6} - \frac{9(10)(35)}{6} \\
&= 10(7)(79) - 3(5)(35) \\
&= 7(790 - 75) \\
&= 5005.
\end{aligned}$$

EXERCISE 3.5

Derive formulae for the evaluation of each of the following series.

1. $2 + 5 + 8 + \cdots + (3n - 1)$
2. $46 + 42 + 38 + \cdots + (50 - 4n)$
3. $\sum_{k=1}^{n} (3k^2 - k)$
4. $\sum_{k=1}^{n} (k^2 + 2k - 1)$
5. $\sum_{k=1}^{n} (2k^3 - k)$
6. $\sum_{i=1}^{n} (i^2 - 3i + 4)$
7. $(1)(3) + (2)(5) + (3)(7) + \cdots + n(2n + 1)$
8. $(2)(8) + (5)(6) + (8)(4) + \cdots + (3n - 1)(10 - 2n)$
9. $(1)(2)(3) + (2)(3)(4) + (3)(4)(5) + \cdots + n(n + 1)(n + 2)$
10. $(1)(2)(2) + (2)(3)(5) + (3)(4)(8) + \cdots + n(n + 1)(3n - 1)$

Evaluate each of the following.

11. $\sum_{k=1}^{15} k^2$
12. $\sum_{p=1}^{20} (p + 3)$
13. $\sum_{i=1}^{30} (3i^2 - i)$
14. $\sum_{k=1}^{18} (2k^2 - 3k + 2)$
15. $\sum_{k=1}^{8} (k^3 + 3k + 5)$
16. $\sum_{i=1}^{12} (4i^3 - 6i^2 + 2i - 3)$
17. $\sum_{k=20}^{35} (2k + 3)$
18. $\sum_{p=15}^{33} (p^2 - 3p + 2)$
19. $\sum_{k=18}^{25} (4k^3 - 3k^2 + 2)$
20. $\sum_{k=40}^{60} (k^2 - 5k)$
21. $\sum_{k=80}^{85} (k^3 + 3k^2 - 10k)$
22. $\sum_{k=86}^{100} (12k^2 - 10k)$
23. Using the identity $n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1$, derive a formula for
$$\sum_{k=1}^{n} k^3.$$
24. Using the methods of this section, derive formulae for
$$\sum_{k=1}^{n} k^4 \quad \text{and} \quad \sum_{k=1}^{n} k^5.$$

3.6. Partial Fractions (Supplementary)

The rational algebraic expressions $\frac{2}{1 + x}$ and $\frac{-1}{1 - 2x}$ may be added to give $\frac{1 - 5x}{1 - x - 2x^2}$. Many series, whose general term is of the form $\frac{P(x)}{Q(x)}$, may be summed by using the reverse procedure of separating the fractional polynomial into its *partial fractions*. The first problem is to find the partial fractions.

Example 1. Separate $\dfrac{1-5x}{1-x-2x^2}$ into partial fractions.

Solution:

$$\frac{1-5x}{1-x-2x^2} = \frac{1-5x}{(1+x)(1-2x)}.$$

Let

$$\begin{aligned}\frac{1-5x}{(1+x)(1-2x)} &= \frac{A}{1+x} + \frac{B}{1-2x}.\\ &= \frac{A(1-2x)+B(1+x)}{(1+x)(1-2x)}.\end{aligned}$$

Therefore

$$1-5x \equiv A(1-2x) + B(1+x).$$

(Note that the identity sign $\equiv$ is used to emphasize that this is an identity and hence true for all $x \in Re$.)

Replace x by $\frac{1}{2}$.

$$\begin{aligned}1 - 2\tfrac{1}{2} &= B(1+\tfrac{1}{2}).\\ -1\tfrac{1}{2} &= B(1\tfrac{1}{2}).\end{aligned}$$

Therefore

$$B = -1.$$

Replace x by -1.

$$1+5 = A(1+2).$$

Therefore

$$A = 2.$$

Therefore

$$\frac{1-5x}{(1+x)(1-2x)} = \frac{2}{1+x} - \frac{1}{1-2x}.$$

When the partial fractions contain only simple linear polynomials as denominators, the process may be quite easily performed mentally. In replacing x by $\frac{1}{2}$, we are making the term $A(1-2x) = 0$. Therefore, replacing x by $\frac{1}{2}$,

$$1-5x = B(1+x),$$

and so

$$B = \frac{1-5x}{1+x}, \quad \text{when } x = \tfrac{1}{2}.$$

Similarly, replacing x by -1,

$$B(1+x) = 0,$$

and so

$$A = \frac{1-5x}{1-2x}, \quad \text{when } x = -1.$$

Hence the numerator for $1-2x$ is the value of $\dfrac{1-5x}{1+x}$ when x is replaced by $\frac{1}{2}$, and the numerator for $1+x$ is the value of $\dfrac{1-5x}{1-2x}$ when x is replaced by -1. It is certainly advisable, however, to perform the complete development as a check, until confidence in performing the mental calculation is acquired.

Example 2. Evaluate $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)}$.

Solution: Let

$$\frac{1}{(3k-2)(3k+1)} = \frac{A}{3k-2} + \frac{B}{3k+1}$$
$$= \frac{A(3k+1) + B(3k-2)}{(3k-2)(3k+1)}.$$

Therefore

$$1 \equiv A(3k+1) + B(3k-2).$$

When k is replaced by $-\frac{1}{3}$,

$$1 = -3B.$$

Therefore

$$B = -\tfrac{1}{3}.$$

When k is replaced by $\frac{2}{3}$,

$$1 = 3A.$$

Therefore

$$A = \tfrac{1}{3}.$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{3(3k-2)} - \frac{1}{3(3k+1)}\right) \\
&= \sum_{k=1}^{n} \frac{1}{3(3k-2)} - \sum_{k=1}^{n} \frac{1}{3(3k+1)} \\
&= \frac{1}{3} \sum_{k=1}^{n} \frac{1}{3k-2} - \frac{1}{3} \sum_{k=1}^{n} \frac{1}{3k+1} \\
&= \frac{1}{3} \sum_{k=1}^{n} \frac{1}{3k-2} - \frac{1}{3} \sum_{k=2}^{n+1} \frac{1}{3(k-1)+1} \qquad (1) \\
&= \frac{1}{3} \sum_{k=1}^{n} \frac{1}{3k-2} - \frac{1}{3} \sum_{k=2}^{n+1} \frac{1}{3k-2} \\
&= \frac{1}{3}\left(\frac{1}{1}\right) - \frac{1}{3}\left(\frac{1}{3(n+1)-2}\right) \qquad (2) \\
&= \frac{1}{3} - \frac{1}{3(3n+1)} \\
&= \frac{3n+1-1}{3(3n+1)}.
\end{aligned}$$

Therefore

$$\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}.$$

In the step marked (1) we are using Summation Property 5 (Section 3.2). The substitution of $k-1$ for k is used to have both summands equal and so make step (2) possible. In step (2) all the terms of each series are equal in value but opposite in sign, with the exception of the term obtained from $k = 1$ in the first series and from $k = n + 1$ in the second. Compare the result with that obtained by mathematical induction in question (9), Exercise 3.4. Although the method given here takes longer, we do not have to know the probable result. The result is produced by the method given here.

EXERCISE 3.6

Resolve the following into partial fractions.

1. $\dfrac{2x+1}{x^2+x-2}$
2. $\dfrac{3}{x^2+x-2}$
3. $\dfrac{3x+1}{x^2-x-6}$
4. $\dfrac{x-4}{2-3x+x^2}$
5. $\dfrac{6+x}{3+x-2x^2}$
6. $\dfrac{2k+5}{4k^2-1}$
7. $\dfrac{1}{9k^2+3k-2}$
8. $\dfrac{5}{25k^2+5k-6}$

Derive a formula to evaluate the sum of each of the following series.

9. $\sum\limits_{k=1}^{n} \dfrac{1}{k^2+k}$
10. $\sum\limits_{i=1}^{n} \dfrac{1}{4i^2-1}$
11. $\sum\limits_{p=1}^{n} \dfrac{4}{(4p-1)(4p+3)}$
12. $\sum\limits_{p=1}^{n} \dfrac{1}{(4p-1)(4p+3)}$
13. $\sum\limits_{i=1}^{n} \dfrac{1}{25i^2+5i-6}$
14. $\sum\limits_{k=1}^{n} \dfrac{3}{9k^2+3k-2}$
15. $\sum\limits_{k=1}^{n} \dfrac{4}{4k^2-4k-3}$
16. $\sum\limits_{k=1}^{n} \dfrac{3}{(3k-4)(3k+2)}$

Chapter Summary

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n\,.$$

Properties of Summation

(1) $$\sum_{i=1}^{n} a = na\,.$$

(2) $$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j\,.$$

(3) $$\sum_{i=1}^{n} a(b_i) = a \sum_{i=1}^{n} b_i .$$

(4) $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i .$$

(5) $$\sum_{i=m}^{n} a_i = \sum_{i=m+p}^{n+p} a_{i-p} .$$

(6) $$\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i .$$

Axiom of Mathematical Induction

A set S of natural numbers is the set N of all natural numbers if

1. $1 \in S$,
2. $k + 1 \in S$ whenever $k \in S$.

Summation Formulae

(1) The sum of the arithmetic series

$$\sum_{k=1}^{n} [a + (k-1)d] = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d]$$

$$= \frac{n}{2}[2a + (n-1)d] .$$

(2) The sum of the geometric series

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \qquad (r \neq 1).$$

(3) $$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} .$$

(4) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} .$$

(5) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} .$$

REVIEW EXERCISE 3

1. Express each of the following series in expanded form.

(a) $\sum_{k=1}^{6} (2k - 3)$

(b) $\sum_{n=4}^{10} (-1)^n (2n)$

(c) $\sum_{i=1}^{8} (4 - i)$

(d) $\sum_{i=-3}^{3} (i^2 + 1)$

2. Express each of the following series in sigma notation.

(a) $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$

(b) $3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35$

(c) $\frac{1}{9} + \frac{1}{3} + 1 + 3 + 9 + 27 + 81$

(d) $\dfrac{1}{a^2 + 2ab + b^2} + \dfrac{1}{a + b} + 1 + (a + b) + (a^2 + 2ab + b^2)$

(e) $\dfrac{1}{(3)(4)} + \dfrac{1}{(4)(5)} + \dfrac{1}{(5)(6)} + \dfrac{1}{(6)(7)} + \dfrac{1}{(7)(8)}$

(f) $\dfrac{1}{(1)(3)} + \dfrac{1}{(2)(5)} + \dfrac{1}{(3)(7)} + \dfrac{1}{(4)(9)} + \cdots + \dfrac{1}{n(2n + 1)}$

3. Express each of the following with 1 as the lower limit of summation.

(a) $\sum_{i=3}^{8} (2i - 1)$ (b) $\sum_{k=-2}^{3} k^2$ (c) $\sum_{k=0}^{n} 2^k$

(d) $\sum_{p=2}^{n+1} (p^2 - 1)$ (e) $\sum_{k=4}^{8} |3 - k|$ (f) $\sum_{i=-1}^{n-2} (-1)^i(3i)$

4. Use the axiom of mathematical induction to prove each of the following.

(a) $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n + 1)}{2}$.

(b) $\dfrac{1}{2} - \dfrac{1}{4} - \dfrac{1}{8} - \cdots - \dfrac{1}{2^n} = \dfrac{1}{2^n}$.

(c) $\sum_{p=1}^{n} \dfrac{1}{p^2 + 3p + 2} = \dfrac{n}{2(n + 2)}$.

(d) $\sum_{k=1}^{n} \dfrac{1}{k(k + 2)} = \dfrac{3n^2 + 5n}{4(n + 1)(n + 2)}$.

(e) $3^n \geq 2n + 1$ for all $n \in N$.

(f) $2^n \geq 2n$ for all $n \in N$.

(g) $3^{2n} - 1$ is divisible by 8 for all $n \in N$.

(h) $n^2(n + 1)^2$ is divisible by 4 for all $n \in N$.

5. Evaluate each of the following.

(a) $\sum_{k=1}^{n} (3k - 1)$ (b) $\sum_{i=1}^{n} (1 + i)$ (c) $\sum_{p=1}^{n} (3p^2 - 2p + 1)$

(d) $\sum_{x=1}^{n} (2x - 1)(x + 1)$ (e) $\sum_{x=1}^{n} (x^3 - 3x + 2)$ (f) $\sum_{k=1}^{n} (4k^3 - 6k^2 + 2k - 3)$

(g) $\sum_{k=1}^{12} (5k - 2)$ (h) $\sum_{k=1}^{15} (2k^2 + k - 4)$ (i) $\sum_{k=1}^{25} (k^2 + k + 1)$

(j) $\sum_{k=20}^{30} (4k + 1)$ (k) $\sum_{x=31}^{40} (3x^2 - x - 2)$ (l) $\sum_{p=11}^{22} (2p^3 - 3p^2 - p + 2)$

6. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all $n \in N$.
Note: $x^{2k+2} - y^{2k+2} = x^{2k+2} - x^2y^{2k} + x^2y^{2k} - y^{2k+2}$.

7. For the sequence defined by $t_1 = 2$, $t_{n+1} = 1 + \sqrt{1 + t_n}$, prove that
   (a) $t_n < 3$,
   (b) $t_{n+1} > t_n$ for all $n \in N$.

8. For the sequence defined by $t_1 = 5$, $t_{n+1} = \sqrt{2t_n + 4}$, prove that
   (a) $t_n > 3$,
   (b) $t_{n+1} < t_n$ for all $n \in N$.

9. (Supplementary) Resolve into partial fractions.
   (a) $\dfrac{5x + 1}{x^2 + x - 2}$ (b) $\dfrac{x - 7}{x^2 + x - 6}$
   (c) $\dfrac{4}{1 - x^2}$ (d) $\dfrac{4}{x^2 + x - 2}$

10. (Supplementary) Evaluate each of the following.
   (a) $\displaystyle\sum_{k=1}^{n} \frac{1}{16k^2 - 4}$ (b) $\displaystyle\sum_{x=1}^{n} \frac{1}{(3x - 1)(3x + 2)}$
   (c) $\displaystyle\sum_{k=1}^{n} \frac{5}{25k^2 + 15k - 4}$ (d) $\displaystyle\sum_{k=2}^{n} \frac{2}{k^2 - 1}$
   (e) $\displaystyle\sum_{k=1}^{n} \frac{4}{4k^2 - 4k - 3}$ (f) $\displaystyle\sum_{i=1}^{10} \frac{6}{(6i - 1)(6i + 5)}$

Chapter 4

THE BINOMIAL THEOREM

4.1. Powers of a Binomial Base

Let us consider the expansion of $(a+b)^3$, using only the axioms of algebra.

$$\begin{aligned}
(a+b)^3 &= (a+b)(a+b)(a+b) && \\
&= (a+b)[(a+b)(a+b)] && \text{Associative Law} \\
&= (a+b)[(a+b)a+(a+b)b] && \text{Distributive Law} \\
&= (a+b)[aa+ba+ab+bb] && \text{(Dist.)} \\
&= a[aa+ba+ab+bb]+b[aa+ba+ab+bb] && \text{(Dist.)} \\
&= \underset{1}{aaa}+\underset{2}{aba}+\underset{3}{aab}+\underset{4}{abb}+\underset{5}{baa}+\underset{6}{bba}+\underset{7}{bab}+\underset{8}{bbb}. && \text{(Dist.)}
\end{aligned}$$

The eight terms (Some, of course, are like terms) have been numbered. Each term is the product of three factors, the first being obtained from the first binomial, the second from the second binomial, and the third from the third binomial in $(a+b)(a+b)(a+b)$. All such possible selections are included in the eight terms. The third term, aab, is formed by selecting a from each of the first two binomials and b from the third. Each term will, therefore, be the product of three letters, one selected from each of the binomial factors. In the expansion, a^3 appears only once (aaa), a^2b appears three times (aba, aab, baa), ab^2 appears three times (abb, bba, bab), and b^3 appears once (bbb). Therefore

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 .$$

We see that each term of this is formed by selecting one and only one term from each of the three binomial factors, multiplying these terms together, and adding all like products. We note that we can select either none, one, two, or three b's. If we select no b's, then we must select one a from each factor. This can be done in only one way and gives the first term, a^3. If we select one b, we can select it from any one of the three factors. This can be done in $\binom{3}{1} = 3$ ways. For each of these, we must select one a from each of the other two factors and this can be done in only one way. Hence, we have three terms, each equal to a^2b, giving the second term, $3a^2b$. We may now select two b's from the three factors in $\binom{3}{2} = 3$ ways, and an a must then be selected in one way from the remaining factor. This gives the third term, $3ab^2$. Finally, we may select three b's (one from each factor) in $\binom{3}{3} = 1$ way and we obtain the fourth term, b^3. Therefore

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 .\end{aligned}$$

We note that this may be written as

$$(a+b)^3 = \sum_{k=0}^{3} \binom{3}{k} a^{3-k}b^k .$$

In the expansion of $(a+b)^6$, each term will be of dimension six in a and b since each term will be a product of one letter from each of the six binomial factors; hence it will be of the form $a^{6-r}b^r$, where $r \in \{0, 1, 2, 3, 4, 5, 6\}$. By using a similar argument to that used in the expansion of $(a+b)^3$, we see that

$$\begin{aligned}
&\text{when } r = 0, \text{ the coefficient of } a^6 \text{ is } \binom{6}{0}, \\
&\text{when } r = 1, \text{ the coefficient of } a^5b \text{ is } \binom{6}{1}, \\
&\text{when } r = 2, \text{ the coefficient of } a^4b^2 \text{ is } \binom{6}{2}, \\
&\text{when } r = 3, \text{ the coefficient of } a^3b^3 \text{ is } \binom{6}{3}, \\
&\text{when } r = 4, \text{ the coefficient of } a^2b^4 \text{ is } \binom{6}{4}, \\
&\text{when } r = 5, \text{ the coefficient of } ab^5 \text{ is } \binom{6}{5}, \\
&\text{when } r = 6, \text{ the coefficient of } b^6 \text{ is } \binom{6}{6}.
\end{aligned}$$

Therefore

$$\begin{aligned}(a+b)^6 &= \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6 \\ &= \sum_{r=0}^{6} \binom{6}{r} a^{6-r}b^r \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 .\end{aligned}$$

EXERCISE 4.1

1. If the numerical coefficients are disregarded, terms of the following form will appear in the expansion of $(a + b)^7$. State the value of the exponent r in each case.

   (a) a^5b^r (b) a^2b^r (c) a^rb^4 (d) a^rb^2

2. In each of the following, write the expansion in sigma-notation form, in r-subset notation form $\binom{n}{r}$, and in fully expanded form.

   (a) $(a + b)^4$ (b) $(a + b)^5$ (c) $(x + y)^8$ (d) $(p + q)^7$

3. How many different terms are there in the expansion of each of the binomials in question (2)?

4. Without expanding, state the number of different terms in the expansions of (a) $(a + b)^{11}$, (b) $(x + y)^{15}$, (c) $(p + q)^{18}$, (d) $(1 + x)^{20}$.

4.2. Expansion of $(a + b)^n$, $n \in N$

$$(a + b)^n = (a + b)(a + b)(a + b) \cdots n \text{ factors.}$$

Each term in the expansion is obtained by forming the product of one term from each binomial factor. Each term of the expansion will, therefore, be of the form

$$a^{n-r}b^r, \quad \text{where } r \in \{0, 1, 2, \ldots, n\}.$$

This general term $a^{n-r}b^r$ is formed by selecting one b from each of r of the n factors. This can be done in $\binom{n}{r}$ ways. For each of these ways, the $(n - r)$ a's can be selected, one from each of the remaining $(n - r)$ binomial factors, in only one way. Hence, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$. Therefore

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$$

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n.$$

We note the following points in this expansion:

(i) The number of different terms is $n + 1$.

(ii) The general term $\binom{n}{r}a^{n-r}b^r$ is the $(r + 1)$st term.

(iii) The coefficients of terms equidistant from the ends of this expansion are equal.

The following expansions were obtained from the previous section or are known from earlier work.

$$
\begin{aligned}
(a+b)^0 &= 1\,.\\
(a+b)^1 &= a+b\,.\\
(a+b)^2 &= a^2+2ab+b^2\,.\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\,.\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\,.\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b\,.\\
(a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\,.
\end{aligned}
$$

It can be seen that the coefficients follow a definite pattern which can be most easily demonstrated by means of Pascal's Triangle.

$$
\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

Can you see how the pattern develops? What will be the next two lines in the triangle? In forming the lines in the triangle, we are making use of the identity which you were asked to prove in question (21), Exercise 2.4.

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}.$$

This identity will be used again in the next section.

Example 1. Express (a) $(a+2b)^4$, (b) $(2a-3b)^7$ using sigma notation.

Solution:

(a)

$$
\begin{aligned}
(a+2b)^4 &= \sum_{r=0}^{4}\binom{4}{r}a^{4-r}(2b)^r\\
&= \sum_{r=0}^{4}\binom{4}{r}2^r a^{4-r}b^r\,.
\end{aligned}
$$

(b)

$$
\begin{aligned}
(2a-3b)^7 &= \sum_{r=0}^{7}\binom{7}{r}(2a)^{7-r}(-3b)^r\\
&= \sum_{r=0}^{7}\binom{7}{r}(-1)^r(2^{7-r})\,(3^r)a^{7-r}b^r\,.
\end{aligned}
$$

Example 2. Expand (a) $(a + 2b)^4$, (b) $(2a - 3b)^7$.

Solution:

$$\text{(a)}\quad (a + 2b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3(2b) + \binom{4}{2}a^2(2b)^2 + \binom{4}{3}a(2b)^3 + \binom{4}{4}(2b)^4$$
$$= a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 .$$

$$\begin{aligned}\text{(b)}\quad (2a - 3b)^7 = {} & \binom{7}{0}(2a)^7 + \binom{7}{1}(2a)^6(-3b) + \binom{7}{2}(2a)^5(-3b)^2 \\ & + \binom{7}{3}(2a)^4(-3b)^3 + \binom{7}{4}(2a)^3(-3b)^4 + \binom{7}{5}(2a)^2(-3b)^5 \\ & + \binom{7}{6}(2a)(-3b)^6 + \binom{7}{7}(-3b)^7 \\ = {} & 128a^7 - 1344a^6b + 6048a^5b^2 - 15120a^4b^3 + 22680a^3b^4 \\ & - 20412a^2b^5 + 10206ab^6 - 2187b^7 .\end{aligned}$$

Example 3. Find the first three terms and the seventeenth term in the expansion of $(x - 2y)^{19}$.

Solution:

$$\begin{aligned}(x - 2y)^{19} &= \binom{19}{0}x^{19} + \binom{19}{1}x^{18}(-2y) + \binom{19}{2}x^{17}(-2y)^2 + \cdots \\ &= x^{19} - 38x^{18}y + 684x^{17}y^2 + \cdots . \\ t_{17} &= \binom{19}{16}x^3(-2y)^{16} \\ &= 2^{16} \cdot 969x^3y^{16} .\end{aligned}$$

Example 4. Write the term containing a^8 in the expansion of $\left(a^2 - \dfrac{b}{2}\right)^6$.

Solution: Since the required term must contain a^8 or $(a^2)^{6-2}$, the coefficient must be $\binom{6}{2}$. Hence

$$\begin{aligned}t_3 &= \binom{6}{2}(a^2)^4\left(-\frac{b}{2}\right)^2 \\ &= \frac{15a^8b^2}{4} .\end{aligned}$$

EXERCISE 4.2

1. Write the following expansions in sigma notation.

   (a) $(x + y)^{12}$ (b) $(x - y)^9$ (c) $(2x + y)^7$

   (d) $\left(2x + \dfrac{y}{2}\right)^8$ (e) $(x^2 + 1)^{10}$ (f) $(1 - 2x^2)^{15}$

2. Expand each of the following.

(a) $(a-b)^5$ (b) $(2a+b)^4$ (c) $\left(2x-\frac{y}{2}\right)^6$

(d) $\left(x^2+\frac{1}{x}\right)^4$ (e) $\left(4x^2-\frac{x}{2}\right)^4$ (f) $\left(3a-\frac{2}{b}\right)^5$

3. Write the first three terms of each of the following expansions.

(a) $(a+b)^{10}$ (b) $(a-b)^8$ (c) $(a+3b)^7$

(d) $(1-3x)^{15}$ (e) $(3a-2)^5$ (f) $(1+2b)^{12}$

(g) $\left(2x-\frac{y}{4}\right)^8$ (h) $\left(x^2-\frac{2}{x}\right)^6$ (i) $\left(3x^2-\frac{x}{y}\right)^6$

4. Find the indicated term in each of the following expansions.

(a) $(a+b)^{10}$, t_7 (b) $(a-2b)^7$, t_4

(c) $\left(x^2+\frac{x}{2}\right)^6$, t_3 (d) $\left(x^2-\frac{2}{x}\right)^{10}$, t_4

(e) $\left(2x-\frac{y}{3}\right)^7$, t_5 (f) $(3x+2y)^6$, middle term

(g) $\left(\frac{2}{a}+\frac{a}{2}\right)^8$, t_5 (h) $\left(\frac{2}{a}+\frac{a}{2}\right)^{2n}$, t_{n+1}

5. Write the term in the expansion of $(x+2y)^7$ in which the exponent of x is 5.

6. Write the term in the expansion of $(2x^2-y)^8$ in which the exponent of x is 6.

7. Write the term in the expansion of $\left(3x^2-\frac{2}{y^2}\right)^6$ in which the exponent of y is -8.

8. In the expansion of $(ax+by)^7$, the coefficients of the first two terms are 128 and -224, respectively. Find the values of a and b.

9. In the expansion of $(ax+by)^n$, the coefficients of the first three terms are 1, $-\frac{16}{3}$, and $\frac{112}{9}$, respectively. If $a>0$, find the values of a, b, and n.

4.3. Proof by Mathematical Induction

The binomial theorem may be proved by mathematical induction, but we must, obviously, obtain a possible solution by "intelligent guessing" before we can attempt a proof by this method. For this reason the proof given in the previous section has the advantage of constructing the expansion as the proof is developed. We note that it is required to prove that

$$(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \qquad (r\in W,\, n\in N)\,.$$

In the proof we require the identity which we were asked to establish in question (21), Exercise 2.4.

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} \qquad (1).$$

We will also use the identities

$$\binom{k}{0} = \binom{k+1}{0} = 1 \text{ and } \binom{k}{k} = \binom{k+1}{k+1} = 1 \qquad (2).$$

Proof: Let S be the set of natural numbers for which

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r .$$

1.
$$\sum_{r=0}^{1} \binom{1}{r} a^{1-r} b^r = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1$$
$$= a + b .$$

Therefore

$$1 \in S .$$

2. Assume that $k \in S$.

Hence

$$\begin{aligned}(a+b)^k &= \sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r \\ &= \binom{k}{0} a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} \\ &\quad + \binom{k}{r} a^{k-r} b^r + \cdots + \binom{k}{k} b^k .\end{aligned}$$

Then

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= a(a+b)^k + b(a+b)^k \\ &= \binom{k}{0} a^{k+1} + \binom{k}{1} a^k b + \binom{k}{2} a^{k-1} b^2 + \cdots \\ &\quad + \binom{k}{r-1} a^{k-r+2} b^{r-1} + \binom{k}{r} a^{k-r+1} b^r + \cdots + \binom{k}{k} a b^k \\ &\quad + \binom{k}{0} a^k b + \binom{k}{1} a^{k-1} b^2 + \binom{k}{2} a^{k-2} b^3 + \cdots \\ &\quad + \binom{k}{r-1} a^{k-r+1} b^r + \binom{k}{r} a^{k-r} b^{r+1} + \cdots + \binom{k}{k} b^{k+1}\end{aligned}$$

$$= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \cdots$$

$$+ \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k-r+1}b^r + \cdots + \binom{k}{k}b^{k+1}$$

$$= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2$$

$$+ \cdots + \binom{k+1}{r}a^{k-r+1}b^r + \binom{k+1}{k+1}b^{k+1}$$

(Here we have used the identities (2) in the first and last terms and identity (1) in the other terms.)

$$= \sum_{r=0}^{k+1}\binom{k+1}{r}a^{k+1-r}b^r .$$

Therefore

$$k + 1 \in S \quad \text{if} \quad k \in S .$$

Hence

$$S = N ,$$

and

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \qquad \text{for all } n \in N .$$

4.4. The General Term

In the expansion of $(a+b)^n$, the general term is $t_{r+1} = \binom{n}{r}a^{n-r}b^r$. Once the general term of any particular expansion has been found, it becomes a relatively simple matter to find any particular term in the expansion.

Example 1. Find, in simplified form, the general term in the expansion of

$$\left(x^2 + \frac{1}{x}\right)^{10} .$$

Solution: In the expansion of $(x^2 + x^{-1})^{10}$,

$$t_{r+1} = \binom{10}{r}(x^2)^{10-r}(x^{-1})^r$$

$$= \binom{10}{r}x^{20-2r}x^{-r}$$

$$= \binom{10}{r}x^{20-3r} .$$

Example 2. Find the coefficients of x^{10}, x^0, and x^{-8} in the expansion of

$$\left(x^3 + \frac{1}{x}\right)^{12}.$$

Solution: In the expansion of $(x^3 + x^{-1})^{12}$,

$$\begin{aligned} t_{r+1} &= \binom{12}{r}(x^3)^{12-r}(x^{-1})^r \\ &= \binom{12}{r}x^{36-3r}x^{-r} \\ &= \binom{12}{r}x^{36-4r}. \end{aligned}$$

(i) If $36 - 4r = 10$, $r = 6\frac{1}{2}$.
Since $r \in W$, there is no term containing x^{10}.

(ii) If $36 - 4r = 0$, $r = 9$.
Hence the coefficient of x^0 is $\binom{12}{9} = 220$.

(iii) If $36 - 4r = -8$, $r = 11$.
Thus the coefficient of x^{-8} is $\binom{12}{11} = 12$.

Note that when $r = 9$, $t_{10} = 220x^0 = 220$, and when $r = 11$, $t_{12} = 12x^{-8}$.

Example 3. In the expansion of $\left(a - \frac{2}{a}\right)^8$, find the terms containing a^4, a^0, $\frac{1}{a^2}$.

Solution: In the expansion of $(a - 2a^{-1})^8$

$$\begin{aligned} t_{r+1} &= \binom{8}{r}a^{8-r}(-2a^{-1})^r \\ &= \binom{8}{r}(-2)^r a^{8-r}a^{-r} \\ &= \binom{8}{r}(-2)^r a^{8-2r}. \end{aligned}$$

(i) When $8 - 2r = 4$, $r = 2$. Therefore

$$\begin{aligned} t_3 &= \binom{8}{2}(-2)^2a^4 \\ &= (28)(4)a^4 \\ &= 112a^4. \end{aligned}$$

(ii) When $8 - 2r = 0$, $r = 4$. Therefore

$$\begin{aligned} t_5 &= \binom{8}{4}(-2)^4 a^0 \\ &= (70)(16) \\ &= 1120\,. \end{aligned}$$

(iii) When $8 - 2r = -2$, $r = 5$. Therefore

$$\begin{aligned} t_6 &= \binom{8}{5}(-2)^5 a^{-2} \\ &= (56)(-32)a^{-2} \\ &= -\frac{1792}{a^2}\,. \end{aligned}$$

EXERCISE 4.3

1. Find the general term in each of the following expansions.
   (a) $(1+x)^{12}$ (b) $(2-a)^8$ (c) $(x^2+x)^7$
   (d) $\left(a+\frac{1}{a}\right)^7$ (e) $\left(p-\frac{1}{p}\right)^8$ (f) $\left(x^2-\frac{1}{x}\right)^{10}$
   (g) $\left(2a+\frac{1}{a^2}\right)^{10}$ (h) $\left(2a-\frac{1}{a^3}\right)^{15}$ (i) $\left(a^3+\frac{2}{a^2}\right)^n$
2. In the expansion of $\left(x+\frac{1}{x^2}\right)^{12}$, find the coefficients of x^9, x^3, x^{-9}.
3. Find the term independent of a in the expansion of $\left(a+\frac{1}{a^4}\right)^{15}$.
4. In the expansion of $\left(2x+\frac{1}{x}\right)^{10}$, find the terms containing x^8, x^0, x^{-4}.
5. In the expansion of $\left(\frac{3x}{2}-\frac{2}{x^2}\right)^6$, find the term independent of x and the term containing x^{-6}.
6. In the expansion of $\left(a^2-\frac{1}{a^3}\right)^9$, find the coefficients of a^3, a^{-7}, a^{-10}.
7. Find the term independent of x in the expansion of $\left(2x+\frac{1}{2\sqrt{x}}\right)^9$.
8. In the expansion of $(x+x^{-1})^n$, the fifth term is independent of x. Find the value of n.
9. In the expansion of $\left(\frac{3x}{2}+\frac{1}{x}\right)^n$, the fourth term contains x^3. Find the value of n.
10. In the expansion of $(1-x)(1+x)^n$, the third term is $35x^2$. Find the value of n.

Chapter Summary

If $n \in N$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n.$$

The general term is

$$t_{r+1} = \binom{n}{r}a^{n-r}b^r.$$

REVIEW EXERCISE 4

1. Write the following expansions using sigma notation.

   (a) $(a+b)^7$ (b) $(x+2y)^n$ (c) $(2a-b)^{10}$

   (d) $\left(x^2 - \frac{1}{x}\right)^8$ (e) $\left(2x^2 + \frac{1}{2x}\right)^{15}$ (f) $(2-x^2)^n$

2. Expand each of the following.

   (a) $(a+b)^6$ (b) $(a-b)^6$ (c) $(2a+3b)^5$

3. Write the first three terms of each of the following expansions.

   (a) $(a+2b)^{12}$ (b) $(2x-y)^8$ (c) $\left(2x^2 + \frac{x}{2}\right)^{10}$

   (d) $\left(3x - \frac{1}{x}\right)^6$ (e) $\left(\frac{x}{2} + \frac{2}{x}\right)^{10}$ (f) $(ax^2+b)^{20}$

4. Find the eleventh term in the expansion of $\left(3x + \frac{1}{x}\right)^{12}$.

5. Find the seventh term in the expansion of $\left(x^2 - \frac{1}{x}\right)^{10}$.

6. In the expansion of $(2x-1)^{10}$, find the term containing x^2.

7. Find the first three terms in the expansion of $(a+1)^8(2a-1)^5$.

8. Find the first three terms in the expansion of $(1-x)^7(2+x)^4$.

9. In the expansion of $\left(3x + \frac{x^2}{3}\right)^{10}$, find the coefficients of x^{12} and x^{15}.

10. Find the term independent of x in the expansion of $\left(2x - \frac{1}{x^2}\right)^6$.

11. Find the middle term in the expansion of $\left(\frac{2}{x} + \frac{x}{2}\right)^{10}$.

12. In the expansion of $(1 + px)^n$, the first three terms are $1 - 36x + 594x^2$. Find the values of n and p.

13. In the expansion of $(1 + px)^n$, the first three terms are $1 + 5x + \frac{45x^2}{4}$. Find the values of n and p.

14. In the expansion of $\left(a^2 - \frac{1}{a^3}\right)^n$, the fifth term is $70a^{-4}$. Find the value of n.

15. In the expansion of $(ax + by)^7$, the coefficients of the first two terms are 128 and -112, respectively. Find the values of a and b.

16. In the expansion of $(ax + by)^n$, the coefficients of the first three terms are 729, 486, and 135, respectively. If $a > 0$, find the values of a, b, and n.

17. If $f(x) = x^n$, find the value of $\frac{f(x + h) - f(x)}{h}$ and the value of

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

18. Write the first five terms in the expansion of $\left(1 + \frac{1}{n}\right)^n$ and find (correct to 1 decimal place) the value of $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

Chapter 5

PROBABILITY

5.1. Introduction and Definitions

Most of us already have some intuitive idea as to what is meant by probability. If we toss a coin, we feel certain that it is just as likely to fall "heads" as to fall "tails"; if we roll a die, we feel that one face is just as likely to end up on top as another. In the case of the coin, we consider that "heads" and "tails" have *equal chances* or are *equally likely outcomes*. With the die, we feel that one dot, two dots, three dots, four dots, five dots, or six dots on top are equally likely outcomes.

If a coin is tossed and we consider heads to be a favourable outcome, then we feel that we have one chance in two of having our wish satisfied. If we consider a three to be a favourable outcome in the toss of a die, then we feel that we have one chance in six of being successful.

In more technical language, we would say that, in the case of the coin, the probability of heads is $\frac{1}{2}$, and, in the case of the die, the probability of three on top is $\frac{1}{6}$. This is usually expressed symbolically as

$$\mathrm{P(head)} = \tfrac{1}{2}\,,$$
$$\mathrm{P(3\ on\ top)} = \tfrac{1}{6}\,.$$

We note, of course, that we are considering only a "fair" coin or die. If the die were more heavily loaded on one side, then the opposite side would be more likely to appear on top. The six faces would no longer be equally likely.

If we consider the probability of drawing an honour card (ace, king, queen, jack, or ten) from a well-shuffled deck of playing cards, we know that we have fifty-two possible outcomes, and of these, twenty are favourable (an honour card). The probability of drawing an honour card is $\frac{20}{52}$. That is,

$$\mathrm{P(honour\ card)} = \tfrac{20}{52} = \tfrac{5}{13}\,.$$

That all outcomes are equally likely is not always true but, for the moment, we will only consider cases where this is true. In the example of drawing an honour card, there are fifty-two equally likely outcomes. Twenty of these outcomes correspond to the event "honour card drawn" and may be considered as favourable outcomes. In the same way, if n is the number of equally likely outcomes of a specific experiment and exactly m of these outcomes ($m \leq n$) correspond to an event A, then the probability of A is m/n. In symbols,

$$\mathrm{P}(A) = \frac{m}{n}.$$

This, of course, does not constitute a rigorous definition of an event or of the probability of an event. It is a useful, intuitive approach and more rigorous definitions will be given in the next section.

Example 1. If one card is drawn at random from a well-shuffled deck of fifty-two playing cards, what is the probability of each of the following events?

(a) Drawing a spade (b) Drawing a king

(c) Drawing a red card (d) Drawing the queen of spades

Solution:

$$\begin{aligned} \mathrm{P}(\text{a spade}) &= \tfrac{13}{52} = \tfrac{1}{4}. \\ \mathrm{P}(\text{a king}) &= \tfrac{4}{52} = \tfrac{1}{13}. \\ \mathrm{P}(\text{red card}) &= \tfrac{26}{52} = \tfrac{1}{2}. \\ \mathrm{P}(\text{queen of spades}) &= \tfrac{1}{52}. \end{aligned}$$

Example 2. A letter is chosen at random from the word *probability*. What is the probability that the letter chosen is

(a) a vowel? (b) a consonant? (c) the letter b? (d) the letter s?

Solution:

$$\begin{aligned} \mathrm{P}(\text{vowel}) &= \tfrac{4}{11}. \\ \mathrm{P}(\text{consonant}) &= \tfrac{7}{11}. \\ \mathrm{P}(b) &= \tfrac{2}{11}. \\ \mathrm{P}(s) &= \tfrac{0}{11} = 0. \end{aligned}$$

Example 3. A group of five men and four women agree that three should be chosen by lot to form a special committee. What is the probability that the committee consists of one man and two women?

Solution: Number of possible committee selections is

$$\begin{aligned} \binom{9}{3} &= \frac{9 \times 8 \times 7}{3 \times 2} \\ &= 84. \end{aligned}$$

Number of possible committee selections consisting of one man and two women is

$$\begin{aligned} \binom{5}{1}\binom{4}{2} &= 5 \times 6 \\ &= 30. \end{aligned}$$

Therefore

$$\text{P(1 man and 2 women)} = \tfrac{30}{84} = \tfrac{5}{14}.$$

Example 4. Twenty tickets are placed in a box and shaken. One ticket is taken from the box and the person whose name is on that ticket receives a prize. What is the probability that Tom wins the prize if his name is on

(a) 4 of the tickets? (b) none of the tickets? (c) all 20 tickets?

Solution:

(a) $\text{P(Tom wins)} = \tfrac{4}{20} = \tfrac{1}{5}.$

(b) $\text{P(Tom wins)} = \tfrac{0}{20} = 0.$

(c) $\text{P(Tom wins)} = \tfrac{20}{20} = 1.$

This last example illustrates that the probability of any event A is a real number between 0 and 1, that is,

$$0 \leq P(A) \leq 1.$$

If the event cannot happen, as in Example 4(b), then the probability is zero. If Tom's name does not appear on any of the tickets, he cannot win the prize.

If $P(A) = 1$, then A is certain. If Tom's name appears on all twenty tickets, then he must win. His success is certain and P(Tom wins) = 1.

EXERCISE 5.1

1. If a die is thrown, what is the probability that the upper face shows
   (a) 4 dots? (b) more than 4 dots?
   (c) fewer than 4 dots? (d) an odd number of dots?
   (e) an even number of dots? (f) more than 3 dots?
2. A bag contains five red balls and three black balls. If one ball is drawn from the bag, what is the probability that it is
   (a) a red ball? (b) a black ball?
3. A letter is chosen at random from the word *Toronto*. What is the probability that the letter chosen is
   (a) the letter *t*? (b) the letter *o*? (c) a consonant?
4. From a group of twenty men and sixteen women, one person is chosen by lot. What is the probability that a man is chosen?
5. From a group of twenty men and sixteen women, two persons are chosen by lot. What is the probability that
   (a) both are men? (b) both are women?
   (c) one man and one woman are chosen?
   What is the sum of the three probabilities?

6. Six slips of paper, numbered 1 to 6, are placed in a bag and two are drawn out. Find the probability of each of the following events.
   (a) The numbers 2 and 5 are drawn.
   (b) The numbers 2 and 5 are drawn in that order.
   (c) Two even numbers are drawn.

7. A three-digit number is formed by drawing three slips from a box containing nine slips of paper numbered 1 to 9. The first digit drawn is to be the hundreds digit, the second is to be the tens digit, and the third is to be the units digit. The slips are not replaced after being drawn. What is the probability that the number formed
   (a) is even? (b) is odd? (c) has units digit 4?

8. Three books are selected at random from eight different books on a shelf. What is the probability that a specific book is included in the selection?

9. From a group of twelve boys, a selection of three will be made by lot to attend the Stanley Cup final. Tom and Jim are very anxious to win. What is the probability that
   (a) both will be selected? (b) only one of them will be selected?

10. If two dice are tossed, what is the probability that
    (a) both upper faces show 6? (b) the upper faces total 4?

11. Assuming that boys and girls have an equal chance of being born, what is the probability that, in a family of three children, all three are girls?

12. If two cards are drawn at random from a well-shuffled deck of fifty-two playing cards, what is the probability of each of the following events?
    (a) Both are clubs. (b) Both are aces.
    (c) Both are red cards. (d) Both are black aces.

5.2. Sample Space

If a nickel and a quarter are tossed together, what is the set of all possible outcomes of the experiment? The answer to this question will depend upon what particular aspect of the experiment interests us. If we are interested simply in whether the coins fall alike (both heads or both tails) or differently (one head and one tail), then we have only two possible outcomes, L (like) and D (different). The set of all possible outcomes is

$$S_1 = \{\mathrm{L}, \mathrm{D}\}.$$

However, we may be interested in the number of heads or the number of tails that show. If we agree to denote x heads and y tails by the ordered pair (x, y), then the set of all possible outcomes is

$$S_2 = \{(2, 0), (1, 1), (0, 2)\}.$$

Any possible outcome of the experiment corresponds to exactly one element of the set S_2.

Again, we may be interested in whether each individual coin falls heads (H) or tails (T). In this case we may list the outcome by the ordered pairs (x, y), where x represents the outcome for the nickel and y, the outcome for the quarter. The set of all possible outcomes is

$$S_3 = \{(H, H), (H, T), (T, H), (T, T)\}.$$

(T, H) means that the nickel fell tails and the quarter fell heads. All possible outcomes are elements of the set S_3.

Each of the sets S_1, S_2, S_3 is a set of all possible outcomes of the experiment. Each set is called a sample space of the experiment. Note that there may be many possible sample spaces of an experiment so that we talk about *a* sample space rather than *the* sample space.

We should also note that S_3 is a more fundamental sample space than either S_1 or S_2. If we know that (T, H) is the actual outcome in S_3, we also know that D is the outcome in S_1, and (1, 1) is the outcome in S_2. However, if we know that L is the outcome in S_1, we do not know the outcome in S_3. It may be (H, H) or (T, T). If we know that the actual outcome in S_2 is (2, 0), then we know that the outcome in S_3 is (H, H); but if the outcome in S_2 is (1, 1), we do not know whether the outcome in S_3 is (H, T) or (T, H).

DEFINITION. A *sample space* of an experiment is a set of elements such that every outcome of the experiment corresponds to exactly one element of the set.

An element in a sample space is often called a sample point. Since the sample space is the set of all possible outcomes, there is a tendency in some texts to use the terms *outcome set* and *outcome* in place of sample space and sample point. There is much to be said in favour of this terminology since by definition a sample space is a set of all possible outcomes. Outcome set would seem to be a more descriptive title. However, although there are good reasons for changing to outcome set, we will continue to use the more traditional title of sample space.

This definition of a sample space now enables us to define an event.

DEFINITION. An *event* is a result of an experiment that may be represented by a subset of a sample space of the experiment.

In a sample space $S_3 = \{(H, H), (H, T), (T, H), (T, T)\}$, the event "both coins fall alike" corresponds to the subset $\{(H, H), (T, T)\}$. The subset in this case contains two elements of the sample space. Other events in this sample space may contain 1, 2, 3, or 4 elements of the sample space.

Example 1. A bag contains twenty red balls and twenty black balls. If three balls are drawn from the bag one after the other without replacement, list two sample spaces for the experiment. How many elements of the most fundamental sample space correspond to the following events?

(a) drawing two red balls and one black ball
(b) drawing three red balls
(c) drawing at least one red ball

Solution:

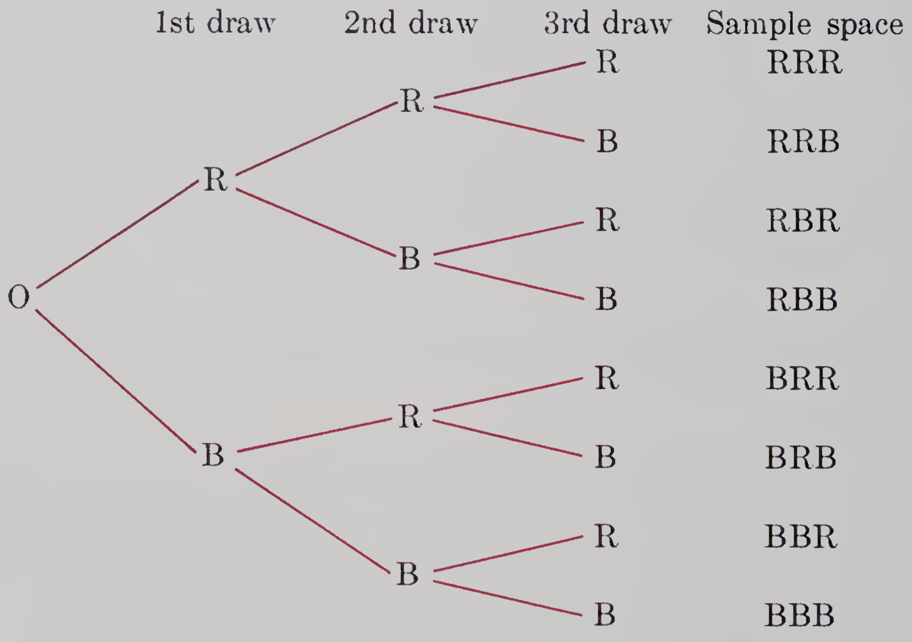
$$S_1 = \{0, 1, 2, 3\},$$

where each element is the number of red balls drawn.

$$S_2 = \{RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB\}.$$

A useful method of setting up S_2 is by means of a tree diagram.

Table 5.1



(a) Three elements of S_2 correspond to drawing two red balls and one black ball.
(b) One element of S_2 corresponds to drawing three red balls.
(c) Seven elements of S_2 correspond to drawing at least one red ball.

Example 2. Two dice, one black and one white, are rolled and the number shown on each die is noted. List a sample space for this experiment.

We note that we are told that the *number* on each die is noted. If only the *total* were required, a suitable sample space would be $\{2, 3, 4, \cdots, 12\}$.

Solution: A tabular arrangement is probably the most suitable way to illustrate the elements of this sample space.

Table 5.2

White die (W)

	W B	1	2	3	4	5	6
Black die B	1	(1, 1)	(1, 2)	(1. 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Note that each row corresponds to a fixed value of the outcome for the black die, and each column, to a fixed value of the outcome for the white die. There are thirty-six elements in the sample space.

Example 3. Using the sample space in Example 2, how many elements of the sample space correspond to the following events?

(a) Both dice show the same number.
(b) The number on the white die is two more than the number on the black die.
(c) The total for the two dice is seven.
(d) The total for the two dice is eleven.

Solution:

(a) Six elements show the same number on both dice.
(b) Four elements correspond to having the number on the white die two more than the number on the black die.
(c) Six elements correspond to a total of seven for the two dice.
(d) Two elements correspond to a total of eleven for the two dice.

EXERCISE 5.2

1. List two sample spaces for an experiment of tossing three coins.
2. Using the sample spaces in question (1), how many elements of each sample space correspond to two heads and one tail?
3. Use a tree diagram to set up a sample space to study the distribution of boys

and girls in families having three children. How many elements of the sample space correspond to

(a) families having three boys?

(b) families having one boy and two girls?

4. Use a tree diagram to set up a sample space to study the distribution of boys and girls in families having four children. How many elements of the sample space correspond to

(a) families having three girls and one boy?

(b) families having two girls and two boys?

(c) families in which the first two children are girls?

5. A box contains five differently coloured balls: red (R), blue (B), green (G), white (W), and yellow (Y). If three have to be chosen, list a sample space of the $\binom{5}{3}$ possible selections. How many elements of the sample space correspond to

(a) a selection including Y?

(b) a selection excluding Y?

(c) a selection including both B and G?

(d) a selection including either B or G?

6. A bag contains twenty blue marbles, twenty green marbles, and twenty red marbles, identical except for colour. Two marbles are drawn one after the other with the first being replaced before the second is drawn. List a sample space for the experiment. How many elements of the sample space correspond to

(a) drawing two marbles of the same colour?

(b) drawing a blue and a green marble?

(c) drawing a blue or a green marble?

7. Four cards, numbered 1, 2, 3, and 4, are to be placed in four boxes numbered 1, 2, 3, 4, one card in each box. List a sample space of ordered quadruples that indicate the card in each slot. (For example, 1243 indicates card 1 in box 1, card 2 in box 2, card 4 in box 3, and card 3 in box 4.) How many elements of the sample space correspond to the following events?

(a) Exactly two card numbers and two box numbers coincide.

(b) At least two card numbers and two box numbers coincide.

(c) No card number coincides with its box number.

5.3. Probabilities in a Finite Sample Space

In Section 5.2, we listed three sample spaces for the example of tossing two coins.

$S_1 = \{L, D\}$, where L indicated that both coins fell heads or both fell tails, and D indicated that one fell heads and the other, tails.

$S_2 = \{(2, 0), (1, 1), (0, 2)\}$, where the ordered pairs (x, y) indicated x heads and y tails.

$S_3 = \{(H, H)\ (H, T), (T, H), (T, T)\}$, where the ordered pairs (x, y) indicated the outcome for the nickel (x) and for the quarter (y).

In Section 5.1, we stated that the probability of an event A is $\frac{m}{n}$, where n is the number of equally likely outcomes and exactly m of these outcomes correspond to event A.

In the experiment of tossing two coins, what is the probability that one falls heads and the other, tails? Using S_1, the outcome corresponding to a favourable event is D, that is, one outcome of the two possible outcomes. From this it would appear that

$$P(1 \text{ head and } 1 \text{ tail}) = \tfrac{1}{2}\,.$$

However, if we consider the three outcomes in S_2 to be equally likely, only the pair (1, 1) corresponds to the favourable event, 1 head and 1 tail. From this it would appear that

$$P(1 \text{ head and } 1 \text{ tail}) = \tfrac{1}{3}\,.$$

From S_3, it would appear that

$$P(1 \text{ head and } 1 \text{ tail}) = \tfrac{2}{4} = \tfrac{1}{2}\,.$$

The difficulty arises because the outcomes in S_2 are not equally likely. If we assume that the outcome of one coin is independent of the outcome of the other, then the four outcomes in S_3 are equally likely and each can be assigned a probability of $\frac{1}{4}$. In this case the outcomes in S_2 are not equally likely and

$$P(2, 0) = \tfrac{1}{4}, \quad P(1, 1) = \tfrac{1}{2}, \quad P(0, 2) = \tfrac{1}{4}\,.$$

It is essential when using a sample space either to ensure that each of the outcomes is equally likely or to know the probability of each possible outcome.

In Table 5.2 of the last section, we set up the thirty-six elements of the sample space for the experiment of rolling two dice, one white and one black, when the number on each die is noted. Naturally we would expect thirty-six elements in this sample space even if they were not listed. If B is the set of all possible outcomes for the black die and W is the set of all possible outcomes for the white die, then

$$\begin{aligned} B &= \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad n(B) = 6\,, \\ W &= \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad n(W) = 6\,. \end{aligned}$$

The set of all possible outcomes of the experiment of rolling both dice is

$$B \times W = \{(b, w) \mid b \in B \text{ and } w \in W\}$$

and

$$\begin{aligned} \mathrm{n}(B \times W) &= \mathrm{n}(B) \times \mathrm{n}(W) \\ &= 6 \times 6 \\ &= 36 . \end{aligned}$$

Since there are thirty-six equally likely outcomes of the experiment of rolling both dice, we assign a probability of $\frac{1}{36}$ to each possible outcome. If the sample space contains n *equally likely* outcomes, we assign probability $1/n$ to each element of the sample space.

An event corresponds to a subset of the sample space. For example, if A is the event that both dice show the same number, then A corresponds to the subset $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ of S, where S is the sample space in Table 5.2. Therefore

$$\mathrm{P}(A) = \tfrac{6}{36} = \tfrac{1}{6} .$$

Of course, it may happen that the elements of the sample space are not equally likely outcomes and we prefer to have a definition of the probability of an event that will cover all cases.

In the two-coin experiment, if $S_3 = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H}), (\mathrm{T}, \mathrm{T})\}$ is the sample space, and if A is the event "at least one coin falls heads", then

$$A = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H})\} \quad \text{and} \quad \mathrm{P}(A) = \tfrac{3}{4} .$$

If we use $S_2 = \{(2, 0), (1, 1), (0, 2)\}$, then

$$A = \{(2, 0), (1, 1)\} .$$

Then

$$\mathrm{P}(2, 0) = \tfrac{1}{4}, \quad \mathrm{P}(1, 1) = \tfrac{1}{2} ,$$

and

$$\mathrm{P}(A) = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4} .$$

To each element of a sample space, we assign a real number called a *probability.* These probabilities must satisfy two conditions.

(1) A probability is a real number x such that $0 \leq x \leq 1$.

(2) The sum of the probabilities assigned to all the elements of any sample space is 1.

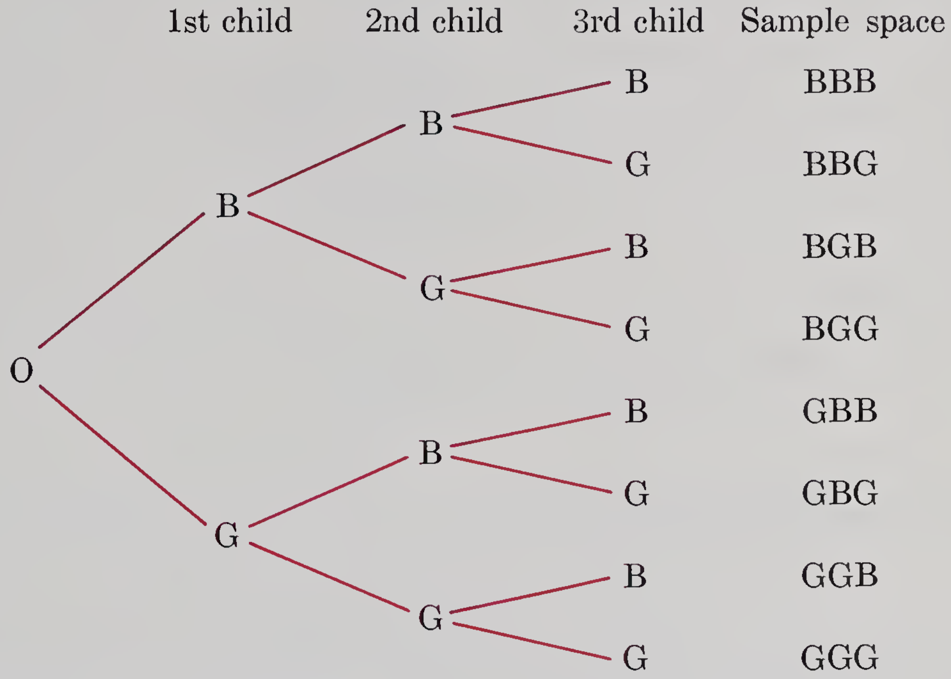
Using this, we may define the probability of an event.

DEFINITION. If A is an event which corresponds to a subset of a sample space, then the probability of A, $\mathrm{P}(A)$, is the sum of the probabilities of the elements of the subset corresponding to A.

Example 1. List the sample space to study the distribution of boys and girls in a family of three children. What is the probability that

(a) all three children are boys?
(b) the first two children are girls?
(c) there are two girls and one boy in a family?

Solution:



There are 8 elements in the sample space.

$$\text{P(3 boys)} = \tfrac{1}{8}\,.$$
$$\text{P(First 2 are girls)} = \tfrac{2}{8} = \tfrac{1}{4}\,.$$
$$\text{P(2 girls, 1 boy)} = \tfrac{3}{8}\,.$$

Note that we are assuming that, when a child is born, it is just as likely to be a boy as a girl. In actual practice, there are slightly more boys born than girls, but for most purposes our assumption is adequate.

Example 2. Using the sample space in Table 5.2 for the two-dice experiment, what is the probability that

(a) the number on the white die is two more than the number on the black die? $(w = b + 2)$
(b) the total for the two dice is seven? $(b + w = 7)$
(c) the total for the two dice is eleven? $(b + w = 11)$

Solution:

$$\mathrm{P}(w = b + 2) = \tfrac{4}{36} = \tfrac{1}{9}\,.$$
$$\mathrm{P}(b + w = 7) = \tfrac{6}{36} = \tfrac{1}{6}\,.$$
$$\mathrm{P}(b + w = 11) = \tfrac{2}{36} = \tfrac{1}{18}\,.$$

Example 3. Using the sample space in Table 5.2 for the two-dice experiment, state the following probabilities.

(a) $\mathrm{P}(A)$, where $A = \{(b, w) \mid b \geq 3\}$

(b) $\mathrm{P}(B)$, where $B = \{(b, w) \mid w \leq 3\}$

(c) $\mathrm{P}(A \text{ and } B)$, where A and B are as described above

(d) $P(A \text{ or } B)$ (Note that *or* is inclusive.)

Solution:

(a) $$\mathrm{P}(A) = \tfrac{24}{36} = \tfrac{2}{3}.$$

(b) $$\mathrm{P}(B) = \tfrac{18}{36} = \tfrac{1}{2}.$$

(c) $$\mathrm{P}(A \text{ and } B) = \tfrac{12}{36} = \tfrac{1}{3}.$$

(d) $$\mathrm{P}(A \text{ or } B) = \tfrac{30}{36} = \tfrac{5}{6}.$$

Note that

$$\begin{aligned}\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \text{ and } B) &= \tfrac{2}{3} + \tfrac{1}{2} - \tfrac{1}{3} \\ &= \tfrac{5}{6} \\ &= \mathrm{P}(A \text{ or } B).\end{aligned}$$

Compare this with the known formula

$$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B).$$

Example 4. A box contains five differently coloured balls: red (R), green (G), blue (B), white (W), and yellow (Y). If a selection of three of the balls is made, state the following probabilities.

(a) The selection includes Y.

(b) The selection excludes Y.

(c) The selection includes both B and G.

(d) The selection includes either B or G.

Solution:

$$S = \{\mathrm{RGB}, \mathrm{RGW}, \mathrm{RGY}, \mathrm{RBW}, \mathrm{RBY}, \mathrm{RWY}, \mathrm{GBW}, \mathrm{GBY}, \mathrm{GWY}, \mathrm{BWY}\}.$$

(a) $$\mathrm{P}(\mathrm{Y} \text{ included}) = \tfrac{6}{10} = \tfrac{3}{5}.$$

(b) $$\mathrm{P}(\mathrm{Y} \text{ excluded}) = \tfrac{4}{10} = \tfrac{2}{5}.$$

(c) $$\mathrm{P}(\mathrm{B} \text{ and } \mathrm{G} \text{ included}) = \tfrac{3}{10}.$$

(d) $$\mathrm{P}(\mathrm{B} \text{ or } \mathrm{G} \text{ included}) = \tfrac{9}{10}.$$

Alternative Solution:

Number of selections without restriction is

$$\binom{5}{3} = 10.$$

Number of selections with Y included is

$$\binom{4}{2} = 6\,.$$

Number of selections with Y excluded is

$$\binom{4}{3} = 4\,.$$

Number of selections with both B and G included is

$$\binom{3}{1} = 3\,.$$

Number of selections with B and G excluded is

$$\binom{3}{3} = 1\,.$$

Therefore the number of selections with B or G included is 9. Hence

$$\begin{aligned} P(\text{Y included}) &= \tfrac{6}{10} = \tfrac{3}{5}\,, \\ P(\text{Y excluded}) &= \tfrac{4}{10} = \tfrac{2}{5}\,, \\ P(\text{B and G included}) &= \tfrac{3}{10}\,, \\ P(\text{B or G included}) &= \tfrac{9}{10}\,. \end{aligned}$$

EXERCISE 5.3

1. Using the sample space given in Table 5.2 for the two-dice experiment, calculate the following probabilities.
   - (a) The numbers on the two dice differ by two.
   - (b) The total is ten or greater.
   - (c) The total is less than seven.
   - (d) The black die shows less than the white die.
   - (e) The total is six.
   - (f) The black die shows less than two.
   - (g) The white die shows greater than four.
   - (h) The black die shows less than two and the white die shows greater than four.
   - (i) The black die shows less than two or the white die shows greater than four.
   - (j) The total on the two dice is a prime number.

(k) If it is known that the total on the two dice is odd, what is the probability that it is seven?

(l) If it is known that the total on the two dice is even, what is the probability that it is seven?

2. In an experiment to investigate the cavity-reducing properties of four different toothpastes, 300 students are divided at random into four equal groups. John and Mary both take part in the test. What is the probability that
   (a) they are both assigned to the same group?
   (b) they are assigned to different groups?
   (c) they are both assigned to group 1?
   (d) John is assigned to either group 1 or 2 and Mary is assigned to either 3 or 4?

3. A coin is tossed three times.
   (a) What is the probability of three heads?
   (b) What is the probability of exactly two heads?
   (c) What is the probability of at least two heads?
   (d) What is the probability of not more than two heads?

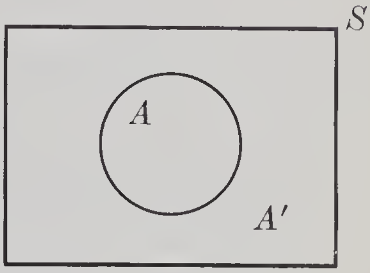
4. Six students, John, Jim, Dick, Mary, Helen, and Betty, have the qualifications for an all-expenses paid trip to New York but, unfortunately, only two can be selected. They agree to make the selection by placing the six names on slips of paper and having the school principal draw two slips at random. What is the probability that
   (a) John is selected?
   (b) both Mary and Betty are selected?
   (c) either Mary or Betty (not both) is selected?
   (d) two girls are selected?

5. List a sample space to study the distribution of boys and girls in families having four children. (See question (4), Exercise 5.2.) What is the probability that
   (a) a family has three boys and one girl?
   (b) a family has two girls and two boys?
   (c) the two eldest children are girls?

6. Four cards numbered 1, 2, 3, and 4 are shuffled and placed in four boxes numbered 1, 2, 3, 4, one card to a box. What is the probability that
   (a) exactly two card numbers and two box numbers coincide?
   (b) at least two card numbers and two box numbers coincide?
   (c) no card number coincides with its box number?
   (d) all card numbers and box numbers coincide?

7. John, Jim, Dick, Mary, and Helen are candidates for three positions on the students' council. Assuming the selection of any three is equally likely, what is the probability that
   (a) Jim is selected?
   (b) Mary and Helen are both selected?
   (c) Dick is not selected?
   (d) either Mary or Helen (or both) is selected?

5.4. Odds



If we consider the sample space in Table 5.2 for the experiment of throwing two dice, we know that it contains thirty-six elements. Any event A will correspond to a subset of the sample space. For any discussion of probabilities in this sample space, the universe of the discussion is the sample space S. If A is the event "throwing a double", the event A is a subset of S.

$$\mathrm{n}(S) = 36 \quad \text{and} \quad \mathrm{n}(A) = 6\,,$$

so that

$$\mathrm{P}(A) = \tfrac{6}{36} = \tfrac{1}{6}\,.$$

The set A' will correspond to the event not-A or "not throwing a double".

$$\mathrm{n}(A') = 30 \quad \text{and} \quad \mathrm{P}(A') = \tfrac{30}{36} = \tfrac{5}{6}\,.$$

In general, if

$$\mathrm{P}(A) = \frac{m}{n}\,,$$

$$\begin{aligned}\mathrm{P}(A') &= \frac{n-m}{n}\\ &= 1 - \frac{m}{n}\\ &= 1 - \mathrm{P}(A)\,.\end{aligned}$$

The relative probabilities of event A and event not-A are frequently expressed in terms of the *odds in favour* of A.

Odds in favour of A are $\mathrm{P}(A)$ to $\mathrm{P}(A')$,

that is,

$$\frac{m}{n} \text{ to } \frac{n-m}{n}\,,$$

or

$$m \text{ to } n - m\,.$$

Also,

$$\text{odds against } A \text{ are } \mathrm{P}(A') \text{ to } \mathrm{P}(A),$$

that is,

$$n - m \text{ to } m.$$

The odds in favour of throwing a double in the two-dice experiment will be given by

$$\mathrm{P}(\text{throwing a double}) \text{ to } \mathrm{P}(\text{not throwing a double}),$$

that is,

$$\frac{1}{6} \text{ to } \frac{5}{6}$$

or

$$1 \text{ to } 5\,.$$

If the odds in favour of an event A are a to b, then

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A')} = \frac{a}{b}\,.$$

Thus

$$\frac{\mathrm{P}(A)}{1 - \mathrm{P}(A)} = \frac{a}{b}$$

$$b \cdot \mathrm{P}(A) = a - a \cdot \mathrm{P}(A)\,.$$

$$(a + b)\mathrm{P}(A) = a\,.$$

Therefore

$$\mathrm{P}(A) = \frac{a}{a + b}\,.$$

Example 1. If two dice are rolled, what are the odds in favour of throwing

(a) 7? (b) 11?

Solution:

$$\mathrm{P}(7) = \frac{6}{36} = \frac{1}{6}\,.$$

Therefore the odds in favour of 7 are 1 to $(6 - 1)$ or 1 to 5.

$$\mathrm{P}(11) = \frac{2}{36} = \frac{1}{18}\,.$$

Therefore the odds in favour of 11 are 1 to 17.

Example 2. If the odds against a certain event are given as 5 to 2, what is the probability of the event?

Solution: Denote the event by A.

$$\begin{aligned} P(A') &= \frac{5}{5+2} = \frac{5}{7}. \\ P(A) &= 1 - P(A') \\ &= 1 - \frac{5}{7} \\ &= \frac{2}{7}. \end{aligned}$$

EXERCISE 5.4

1. A bag contains seven red balls and four black balls. If one ball is drawn from the bag, what are the odds in favour of its being
   (a) a red ball? (b) a black ball?
2. If a die is thrown, what are the odds in favour of the upper face showing
   (a) 4? (b) more than 4?
   What are the odds against the upper face showing six?
3. If one letter is chosen at random from the word *Toronto*, what are the odds in favour of its being the letter *o*? What are the odds against its being the letter *t*?
4. A committee of three is to be chosen by lot from seven candidates. What are the odds that a particular candidate not be selected?
5. From a group of eight men and five women, two are chosen by lot. What are the odds in favour of both chosen individuals being men?
6. If two coins are tossed, what are the odds in favour of both showing heads?
7. In a family of three children, what are the odds against all three being girls?
8. Two cards are drawn at random from a deck of fifty-two playing cards. What are the odds
   (a) that both cards are red?
   (b) against drawing a king?
   (c) in favour of drawing two aces?
9. If two dice are rolled, what are the odds
   (a) in favour of a total of nine?
   (b) against a total of six?
   (c) in favour of a total of more than seven?
   (d) against a total of less than five?

10. Eight men enter a doubles tennis tournament in which partners are selected by lot. What are the odds against having two specified men, A and B, selected as partners?

11. A committee of five is selected by lot from a group of six men and four women. What are the odds in favour of having a specified man and a specified woman on the committee?

12. A plant breeder crosses two plants each possessing the gene pair Aa. Each parent contributes either A or a to the offspring, where the two are combined. What are the odds in favour of the offspring's being of the same genetic type as the parents, that is, Aa?

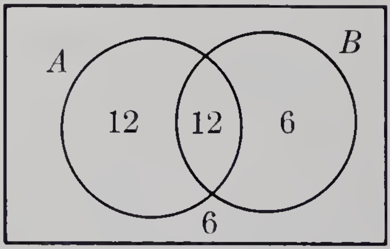
5.5. Addition of Probabilities

In Example 3 of Section 5.3 we discussed the following problem.

Using the sample space given in Table 5.2 for the two-dice experiment, state the following probabilities.

(a) $\mathrm{P}(A)$, where $A = \{(b, w) \mid b \geq 3\}$
(b) $\mathrm{P}(B)$, where $B = \{(b, w) \mid w \leq 3\}$
(c) $\mathrm{P}(A \text{ and } B)$, where A and B are described above
(d) $\mathrm{P}(A \text{ or } B)$

We note that A and B are subsets of the sample space, and that the set *A and B* may be written in set notation as $A \cap B$, while the set *A or B* may be written as $A \cup B$.



In this particular example, $\mathrm{n}(S) = 36$, where S is the sample space, $\mathrm{n}(A) = 24$, $\mathrm{n}(B) = 18$, and $\mathrm{n}(A \cap B) = 12$ as shown in the Venn diagram. We note from the Venn diagram that, when we total the number of elements in the sets A and B, the elements in $A \cap B$ have been counted twice. These twelve elements in this example must then be subtracted in order to obtain the correct number of elements in $A \cup B$. Therefore

$$\mathrm{P}(A) = \tfrac{24}{36} = \tfrac{2}{3}.$$

$$\mathrm{P}(B) = \tfrac{18}{36} = \tfrac{1}{2}.$$

$$\mathrm{P}(A \cap B) = \tfrac{12}{36} = \tfrac{1}{3}.$$

$$\mathrm{P}(A \cup B) = \tfrac{30}{36} = \tfrac{5}{6}.$$

And in this case

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B).$$

This result is true in general and, provided that the sample space consists of *equally likely outcomes*, we may prove it by using a result from Chapter 1.

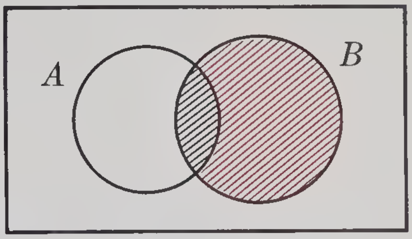
$$\begin{aligned}
\mathrm{n}(A \cup B) &= \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B)\,. \\
\mathrm{P}(A \cup B) &= \frac{\mathrm{n}(A \cup B)}{\mathrm{n}(S)} \\
&= \frac{\mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B)}{\mathrm{n}(S)} \\
&= \frac{\mathrm{n}(A)}{\mathrm{n}(S)} + \frac{\mathrm{n}(B)}{\mathrm{n}(S)} - \frac{\mathrm{n}(A \cap B)}{\mathrm{n}(S)} \\
&= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.
\end{aligned}$$

Therefore

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.$$

We should note that this proof is valid only if the sample space consists of equally likely outcomes, since $\mathrm{P}(A \cup B) = \dfrac{\mathrm{n}(A \cup B)}{\mathrm{n}(S)}$ only in that case. An alternative proof which does not depend on this restriction is now given.

In the Venn diagram, the subset shaded in red is the set $A' \cap B$. The set $A \cup B$ can be expressed as the union of the disjoint subsets A and $A' \cap B$ while the set B can be expressed as the union of the disjoint subsets $A \cap B$ and $A' \cap B$. Then, by definition,



$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)\,.$$

But

$$\mathrm{P}(B) = \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)\,.$$

Thus

$$\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.$$

Therefore

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.$$

Example 1. A box contains five differently coloured balls: red (R), green (G), blue (B), white (W), and yellow (Y). If a selection of three of the balls is made, state the probabilities of each of the following.

(a) The selection includes either Y or G.

(b) The selection includes Y or both B and G.

Solution: Total number of selections without restriction is

$$\binom{5}{3} = 10\,.$$

Number of selections which include Y is

$$\binom{4}{2} = 6\,.$$

Number of selections which include G is

$$\binom{4}{2} = 6\,.$$

Number of selections which include both Y and G is

$$\binom{3}{1} = 3\,.$$

Number of selections which include both B and G is

$$\binom{3}{1} = 3\,.$$

Number of selections which include B, G, and Y is 1.

(a)
$$\begin{aligned} P(Y \cup G) &= P(Y) + P(G) - P(Y \cap G) \\ &= \tfrac{6}{10} + \tfrac{6}{10} - \tfrac{3}{10} \\ &= \tfrac{9}{10}\,. \end{aligned}$$

(b)
$$\begin{aligned} P[Y \cup (B \cap G)] &= P(Y) + P(B \cap G) - P(Y \cap B \cap G) \\ &= \tfrac{6}{10} + \tfrac{3}{10} - \tfrac{1}{10} \\ &= \tfrac{4}{5}\,. \end{aligned}$$

It is suggested that this example should also be solved by forming a sample space as in Example 4 of Section 5.3.

Example 2. One number, x, is selected at random from the set $\{1, 2, 3, \cdots, 10\}$. What is the probability that

(a) $x < 4$?
(b) x is odd?
(c) $x < 4$ or x is odd.

Solution:

(a)
$$P(x < 4) = \tfrac{3}{10}\,.$$

(b)
$$P(x \text{ is odd}) = \tfrac{5}{10} = \tfrac{1}{2}\,.$$

(c)
$$\begin{aligned} P(x < 4 \text{ and } x \text{ is odd}) &= \tfrac{2}{10} = \tfrac{1}{5}\,. \\ P(x < 4 \text{ or } x \text{ is odd}) &= \tfrac{3}{10} + \tfrac{5}{10} - \tfrac{2}{10} \\ &= \tfrac{3}{5}. \end{aligned}$$

EXERCISE 5.5

1. Using the sample space in Table 5.2 for the two-dice experiment, calculate the following probabilities, where b is the number on the black die and w is the number on the white die.
   (a) $b < 3$ and $w > 4$.
   (b) $b < 3$ or $w > 4$.
   (c) $b < 4$ or $w < 4$.
   (d) $b + w = 7$ or $b + w = 11$.

2. One number is selected at random from the set $\{1, 2, 3, \cdots, 20\}$. If x is the number selected, calculate the following probabilities.
   (a) $P(x > 15)$
   (b) $P(A)$, where $A = \{x \mid x \text{ is even}\}$
   (c) $P(B)$, where $B = \{x \mid x < 12\}$
   (d) $P(A \cup B)$

3. If a bag contains four white balls, five red balls, six green balls, and four yellow balls, and one ball is drawn from the bag at random, what is the probability that
   (a) a red ball is selected?
   (b) the ball selected is either red or green?

4. A committee of three is selected by lot from four men and six women. What is the probability that either Mrs. Smith or Mr. Jones is selected? (Note that we always assume the inclusive *or* unless otherwise stated.)

5. John has been given permission to select any three books from a collection of six books {A, B, C, D, E, F}. Calculate the following probabilities.
   (a) The selection includes A.
   (b) The selection includes either A or B.
   (c) The selection includes C or both A and B.
   (d) The selection includes both A and C or both E and F.
   (e) The selection includes both A and C or both C and D.

6. The four aces and four kings are removed from a pack of cards. If two cards are drawn at random from these eight cards, what is the probability that they are both aces or both red?

7. If two cards are drawn at random from a pack of fifty-two cards, what is the probability that they are both aces or both red?

8. There are three tickets available for an N.H.L. game and ten boys agree to draw lots to decide which three will receive the tickets. Calculate each of the following probabilities.
   (a) Either Tom or Jim will be included.

(b) Either Tom and Jim or Jim and Ken will be included.

(c) Either Tom and Jim or Ted and Ken will be included.

9. In a certain school, 15% of the students failed Mathematics, 12% failed English, and 5% failed both Mathematics and English. What is the probability that a particular student, selected at random, failed either English or Mathematics? If there were 550 students who wrote both examinations, how many failed at least one of these subjects?

10. The odds against John's winning a race are 3 to 1 and the odds against Bill's winning are 4 to 1. What is the probability that either Bill or John will win? (Note that a tie is excluded.)

5.6. Mutually Exclusive Events

In question (1d) of Exercise 5.5, we were asked to find the probability that $b + w = 7$ or $b + w = 11$, using Table 5.2 for the two-dice experiment. If $A = \{(b, w) \mid b + w = 7\}$ and $B = \{(b, w) \mid b + w = 11\}$, then $\mathrm{n}(A) = 6$ and $\mathrm{n}(B) = 2$. It is obvious that $A \cap B = \phi$ and $\mathrm{n}(A \cap B) = 0$. Hence

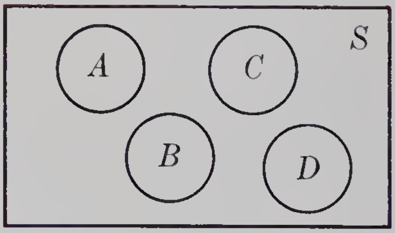
$$\begin{aligned} \mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \\ &= \tfrac{6}{36} + \tfrac{2}{36} - \tfrac{0}{36} \\ &= \tfrac{2}{9}. \end{aligned}$$

In this instance, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

When two events in a given sample space have no elements in common, and their intersection is the empty set, they are said to be mutually exclusive events.

DEFINITION. If two events in a given sample space have no elements in common, they are *mutually exclusive events* or disjoint events.

This definition can be extended to cover any number of events in a given sample space. n events are mutually exclusive if no two of them have any elements in common. In the Venn diagram, the events A, B, C, and D are mutually exclusive.



Theorem. If two events A and B are mutually exclusive, then

$$\mathrm{P}(A \cap B) = 0$$

and

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B).$$

Proof: If A and B are mutually exclusive, then

$$A \cap B = \phi .$$

Therefore

$$\mathrm{n}(A \cap B) = 0$$

and

$$\mathrm{P}(A \cap B) = 0.$$

Therefore

$$\begin{aligned} \mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \\ &= \mathrm{P}(A) + \mathrm{P}(B) . \end{aligned}$$

Corollary. If A, B, and C are mutually exclusive events, then

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) .$$

It must be noted that this only applies if the events are known to be mutually exclusive. If the events do have elements in common and all outcomes are equally likely, we should be able to prove, using the formula for $\mathrm{n}(A \cup B \cup C)$ developed in Chapter 1, that

$$\mathrm{P}(A \cup B \cup C) = \\ \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(A \cap C) + \mathrm{P}(A \cap B \cap C).$$

EXERCISE 5.6

1. In an experiment of throwing two dice, one white and one black, state which of the following events are mutually exclusive.
   (a) The total is seven. Black die shows four.
   (b) The total is five. White die shows six.
   (c) White die shows five. Black die shows two.
   (d) White die shows greater than four. Black die shows less than four.
   (e) White die shows greater than four. Total is less than five.
2. If two coins are tossed, which of the following events are mutually exclusive?
   (a) One coin shows heads. Both coins show heads.
   (b) One coin shows heads. Both coins show tails.
   (c) One coin shows tails. The coins fall differently.
   (d) One coin shows heads. The coins fall alike.
3. In a family of three children, which of the following events are mutually exclusive?
   (a) The two eldest are girls. All three are of the same sex.
   (b) The eldest is a boy. The family has two girls.

(c) Two children are boys. The youngest is a boy.
(d) Two children are boys. The eldest and youngest are girls.
(e) All three are girls. The youngest is a boy.

4. In rolling a single die, A is the event "die shows 3", and B is the event "die shows an odd number". Are A and B mutually exclusive? What is $P(A \cup B)$?

5. In rolling a single die, A is the event "die shows 3", and B is the event "die shows an even number". Are A and B mutually exclusive? What is $P(A \cup B)$?

6. In an experiment of rolling two dice, b represents the number on the black die and w, the number on the white die. Find $P(A \cup B)$ for each of the following events A and B.
   (a) $A = \{(b, w) \mid b + w = 7\}$, $B = \{(b, w) \mid b = 4\}$.
   (b) $A = \{(b, w) \mid b + w = 5\}$, $B = \{(b, w) \mid w = 6\}$.
   (c) $A = \{(b, w) \mid b + w < 5\}$, $B = \{(b, w) \mid w > 4\}$.
   (d) $A = \{(b, w) \mid w = 5\}$, $B = \{(b, w) \mid b = 2\}$.
   (e) $A = \{(b, w) \mid w > 4\}$, $B = \{(b, w) \mid b < 4\}$.
   (f) $A = \{(b, w) \mid b \text{ is odd}\}$, $B = \{(b, w) \mid w = b + 1\}$.

7. If two coins are tossed, find the probability of each of the following.
   (a) One coin shows heads or both coins show heads.
   (b) One coin shows heads or both coins show tails.
   (c) One coin shows tails or the coins fall differently.
   (d) One coin shows heads or both coins fall alike.

8. Find the probability of the following distributions in families of three children.
   (a) The two eldest are girls or all are of the same sex.
   (b) The eldest is a boy or the family has exactly two girls.
   (c) Exactly two children are boys or the youngest is a boy.
   (d) Exactly two children are boys or the eldest and youngest are girls.
   (e) All three are girls or the youngest is a boy.

9. Two cards are drawn from a pack of fifty-two playing cards. Find the probability of each of the following events.
   (a) Both cards are clubs.
   (b) Both cards are black.
   (c) Both cards are kings.
   (d) Both cards are black kings.
   (e) Both cards are kings or both cards are black.

10. Two cards are drawn from a pack of fifty-two playing cards. What is the probability that the cards drawn are the ace and king of spades or that they are both diamonds?

5.7. Independent Events

If, in considering the distribution of children in families, we understand that BGB means that the eldest child is a boy, the second is a girl, and the youngest is a boy, then a sample space, S, of all possible distributions is

$$S = \{\text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}\}.$$

If X is the event "eldest child is a boy", and Y is the event "second child is a girl", then

$$\begin{aligned} \mathrm{P}(X) &= \tfrac{4}{8} = \tfrac{1}{2}, \\ \mathrm{P}(Y) &= \tfrac{4}{8} = \tfrac{1}{2}, \\ \mathrm{P}(X \cap Y) &= \tfrac{2}{8} = \tfrac{1}{4}. \end{aligned}$$

Here we notice that

$$\mathrm{P}(X \cap Y) = \mathrm{P}(X) \cdot \mathrm{P}(Y).$$

If Z is the event "all three are boys", then

$$\mathrm{P}(Z) = \tfrac{1}{8}$$

and

$$\mathrm{P}(X \cap Z) = \tfrac{1}{8}.$$

In this case

$$\mathrm{P}(X \cap Z) \neq \mathrm{P}(X) \cdot \mathrm{P}(Z).$$

We note also that

$$\mathrm{P}(Y \cap Z) = 0$$

and

$$\mathrm{P}(Y \cap Z) \neq \mathrm{P}(Y) \cdot \mathrm{P}(Z).$$

Intuitively, we feel that events X and Y are independent of each other. The sex of the eldest child has no effect on the sex of the second. These events are independent. The events X and Z cannot be regarded as independent. If all three children are boys, such an event does depend on the eldest's being a boy. The two events are dependent. Events Y and Z are, by definition, mutually exclusive. It is impossible for all three children to be boys and the second child to be a girl.

Event X is one that we expect to occur in about one half of all births of eldest children. About one half of all three-child families will have a boy as the eldest child. Restricting ourselves to this one half, we would expect that one half of this group would have a girl as the second child. The fraction of three-child families having a boy as the eldest and a girl as second, would therefore be expected to be about $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$.

Our intuition agrees with the mathematical result in these cases. Independent events are events which have no effect on each other. However, the rather vague statement "having no effect on each other", is not suitable as a mathematical definition.

The results of our example lead us to a definition which satisfies the requirements of mathematical rigour.

DEFINITION. Two events, A and B, are *independent* if and only if

$$P(A \cap B) = P(A) \cdot P(B)\,.$$

Note that this allows us to state that, if two events are such that

$$P(A \cap B) = P(A) \cdot P(B)\,,$$

then we can say that the events are independent. Conversely, if we know that two events are independent, we can state that

$$P(A \cap B) = P(A) \cdot P(B)\,.$$

Note: Confusion sometimes arises between mutually exclusive events and independent events. This should not happen if we remember that, in general, *mutually exclusive events can never be independent.* This can only occur for two events A and B if $P(A) = 0$ or $P(B) = 0$. For two events A and B to be mutually exclusive, $P(A \cap B) = 0$. But, for two events to be independent $P(A \cap B) = P(A) \cdot P(B)$ can only be zero if either $P(A) = 0$ or $P(B) = 0$. If A and B are independent events with *nonzero* probabilities, then $P(A \cap B) \neq 0$, but if the events are mutually exclusive, $P(A \cap B) = 0$.

DEFINITION. Two events, A and B, are dependent if and only if

$$P(A \cap B) \neq P(A) \cdot P(B) \quad \text{and} \quad P(A \cap B) \neq 0.$$

In the example of the three-child family,

$$P(X \cap Y) = P(X) \cdot P(Y)$$

and events X and Y are independent.

$$P(X \cap Z) \neq P(X) \cdot P(Z) \quad \text{and} \quad P(X \cap Z) \neq 0,$$

and events X and Z are dependent.

$$P(Y \cap Z) = 0 \neq P(Y) \cdot P(Z)$$

and events Y and Z are mutually exclusive.

When three or more events are independent, the probability that all occur is the product of their probabilities.

For example, if A, B, and C are independent events, then

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)\,.$$

Example 1. In the two-dice experiment (See Table 5.2.), where b represents the number showing on the black die and w, the number showing on the white die, state which of the following pairs of events are independent, dependent, or mutually exclusive.

(a) $b = 3$, $w = 4$.
(b) $b + w = 7$, $b = 3$.
(c) $b + w = 7$, $b < 3$.
(d) $b + w = 7$, $b \neq 3$.
(e) $b + w = 11$, $b \neq 5$.
(f) $b + w = 11$, $b + w = 7$.

Solution:

(a) Let A be the event $b = 3$ and B, the event $w = 4$.

$$P(A) = \tfrac{6}{36} = \tfrac{1}{6}\,.$$
$$P(B) = \tfrac{6}{36} = \tfrac{1}{6}\,.$$
$$P(A \cap B) = \tfrac{1}{36}\,.$$

Therefore $P(A \cap B) = P(A) \cdot P(B)$ and events A and B are independent.

(b) Let C be the event $b + w = 7$ and D, the event $b = 3$.

$$P(C) = \tfrac{6}{36} = \tfrac{1}{6} \quad \text{and} \quad P(D) = \tfrac{6}{36} = \tfrac{1}{6}\,.$$
$$P(C \cap D) = \tfrac{1}{36}\,.$$

Therefore $P(C \cap D) = P(C) \cdot P(D)$ and events C and D are independent.

(c) Let C be the event $b + w = 7$ and E, the event $b < 3$.

$$P(C) = \tfrac{1}{6} \quad \text{and} \quad P(E) = \tfrac{12}{36} = \tfrac{1}{3}\,.$$
$$P(C \cap E) = \tfrac{2}{36} = \tfrac{1}{18}\,.$$

Therefore $P(C \cap E) = P(C) \cdot P(E)$ and events C and E are independent.

(d) Let C be the event $b + w = 7$ and F, the event $b \neq 3$.

$$P(C) = \tfrac{1}{6} \quad \text{and} \quad P(F) = \tfrac{30}{36} = \tfrac{5}{6}\,.$$
$$P(C \cap F) = \tfrac{5}{36}\,.$$

Therefore $P(C \cap F) = P(C) \cdot P(F)$ and events C and F are independent.

(e) Let G be the event $b + w = 11$ and H, the event $b \neq 5$.

$$P(G) = \tfrac{2}{36} = \tfrac{1}{18} \quad \text{and} \quad P(H) = \tfrac{30}{36} = \tfrac{5}{6}.$$
$$P(G \cap H) = \tfrac{1}{36}.$$

Therefore $P(G \cap H) \neq P(G) \cdot P(H)$ and events G and H are dependent.

(f) Let G be the event $b + w = 11$ and C, the event $b + w = 7$.

$$P(G) = \tfrac{1}{18} \quad \text{and} \quad P(C) = \tfrac{1}{6}.$$
$$P(G \cap C) = 0.$$

Events G and C are mutually exclusive.

Example 2. A bag contains twenty red balls and ten black balls. If one ball is drawn from the bag and returned, and then a second ball is drawn from the bag, what is the probability that the first ball drawn is red and the second, black? (Since the first ball is returned, the events are independent.)

Solution:

$$\begin{aligned} P(\text{red ball}) &= \tfrac{20}{30} = \tfrac{2}{3}. \\ P(\text{black ball}) &= \tfrac{10}{30} = \tfrac{1}{3}. \\ P(\text{red ball followed by black ball}) &= \tfrac{2}{3} \times \tfrac{1}{3} \\ &= \tfrac{2}{9}. \end{aligned}$$

EXERCISE 5.7

1. In the two-dice experiment (see Table 5.2), state whether the following pairs of events are independent, dependent, or mutually exclusive. b represents the number showing on the black die and w, the number showing on the white die.
   (a) $b = 6$, $w = 6$.
   (b) $w < 3$, $b + w = 8$.
   (c) $w < 2$, $b + w = 8$.
   (d) $w \neq 5$, $b + w = 8$.
   (e) $b = 3$ or $w = 3$, $b = 5$.
   (f) $b = 3$ or $w = 3$, $b \neq 5$.

2. A die is thrown three times. Assuming that the throws are independent events, what is the probability that the first throw shows a six, the second shows an even number, and the third shows a five?

3. A coin is tossed six times. Each throw is independent of the other throws. What is the probability of six heads?

4. A bag contains twenty red balls and ten black balls. One ball is drawn from the bag and returned. A second ball is then drawn from the bag. Assuming that the outcomes are independent events, calculate the following probabilities.
   (a) Both balls are red.
   (b) Both balls are black.
   (c) One ball is red and the other is black. (Order is not considered.)
5. Three coins are tossed. Are the following events independent, dependent, or mutually exclusive?
   (a) Heads on first two coins, tails on third coin
   (b) Heads on first two coins, tails on second two coins
   (c) Heads on first two coins, heads on all three coins
6. A bag contains six black balls, four red balls, and five white balls. Three balls are drawn in succession, each one being replaced before the next is drawn. Calculate the following probabilities.
   (a) First is black, second is red, third is white.
   (b) All three are black.
   (c) All three are white.
   (d) First two are black and the third is red.
7. In a family of three children, calculate the following probabilities.
   (a) First is a boy and next two are girls.
   (b) First two are boys and the third is a girl.
   (c) First two are boys or the third is a girl.
8. In a certain school, 15% of the students failed Mathematics, 12% failed English, and 5% failed both English and Mathematics. Are the events "student failed English" and "student failed Mathematics", independent?
9. If two cards are drawn at random from a pack of fifty-two cards, are the events "both cards aces" and "both cards red", independent?
10. If two cards are drawn at random from a pack of fifty-two cards, are the events "first card an ace" and "second card a king", independent? Assume that the first card is returned to the pack before the second is drawn.

5.8. Binomial Distribution

If a bag contains twenty red balls and ten black balls, and six balls are drawn at random with the ball drawn being replaced and the bag well shaken before the next ball is drawn, what is the probability that exactly two red balls are drawn?

Here we have a case of six independent events in which P(red ball) = $\frac{2}{3}$ and P(black ball) = $\frac{1}{3}$ for each event. If, in drawing six balls, we obtain two red balls and four black balls, we have a successful event. Any other combination is not successful.

One successful event is RRBBBB, where R represents a red ball and B, a black ball. The probability of this event is

$$(\tfrac{2}{3})(\tfrac{2}{3})(\tfrac{1}{3})(\tfrac{1}{3})(\tfrac{1}{3})(\tfrac{1}{3}) = (\tfrac{1}{3})^4(\tfrac{2}{3})^2 .$$

However, if a red ball occupies any two of the six positions, we have a successful event. Each of these successful events will have the same probability and all such events are mutually exclusive. The number of such events is $\binom{6}{2}$ and so the probability of exactly two red balls is

$$\begin{aligned}\binom{6}{2}(\tfrac{1}{3})^4(\tfrac{2}{3})^2 &= 15 \times \tfrac{1}{81} \times \tfrac{4}{9}\\ &= \tfrac{20}{243} .\end{aligned}$$

In the same way, if there are n independent events for which the probability of success in any one event is p and the probability of failure is $q = 1 - p$, that is, there are only two possible outcomes for each event, then the probability for exactly r successes in the n events is

$$\binom{n}{r} q^{n-r} p^r .$$

This is the $(r + 1)$st term of the binomial expansion of $(q + p)^n$.

In the example of twenty red balls and ten black balls, where we were seeking the probability of drawing exactly two red balls in six trials;

$$p = \tfrac{2}{3}, \quad q = \tfrac{1}{3}, \quad n = 6 .$$

Therefore

$$\begin{aligned}(q + p)^n &= [(\tfrac{1}{3}) + (\tfrac{2}{3})]^6\\ &= \binom{6}{0}(\tfrac{1}{3})^6(\tfrac{2}{3})^0 + \binom{6}{1}(\tfrac{1}{3})^5(\tfrac{2}{3}) + \binom{6}{2}(\tfrac{1}{3})^4(\tfrac{2}{3})^2 + \binom{6}{3}(\tfrac{1}{3})^3(\tfrac{2}{3})^3 +\\ &\quad \binom{6}{4}(\tfrac{1}{3})^2(\tfrac{2}{3})^4 + \binom{6}{5}(\tfrac{1}{3})(\tfrac{2}{3})^5 + \binom{6}{6}(\tfrac{1}{3})^0(\tfrac{2}{3})^6\\ &= \tfrac{1}{729} + \tfrac{4}{243} + \tfrac{20}{243} + \tfrac{160}{729} + \tfrac{80}{243} + \tfrac{64}{243} + \tfrac{64}{729} .\end{aligned}$$

The successive terms give the probabilities of drawing 0, 1, 2, 3, 4, 5, or 6 red balls, respectively, in any drawing of six balls with replacement between each drawing. It must be emphasized that the events *have to be independent.*

If $P(x)$ represents the drawing of x red balls in any drawing of six balls, then

$$P(0) = \tfrac{1}{729}, \quad P(1) = \tfrac{4}{243}, \quad P(2) = \tfrac{20}{243},$$
$$P(3) = \tfrac{160}{729}, \quad P(4) = \tfrac{80}{243}, \quad P(5) = \tfrac{64}{243}, \quad P(6) = \tfrac{64}{729} .$$

We note that the outcomes, "no red balls", "one red ball", "two red balls", etc., are not equally likely outcomes. Also

$$P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1\,.$$

This should have been expected since these results give all possible results. Also $q+p=1$ and hence $(q+p)^n=1$.

Example 1. If a bag contains twenty red balls and ten black balls and six balls are drawn at random with the ball drawn being replaced before the next ball is drawn, what is the probability that at least three red balls are drawn?

Solution 1:

$$\begin{aligned}
P(3) &= \binom{6}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^3 = \tfrac{160}{729}\,.\\
P(4) &= \binom{6}{4}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^4 = \tfrac{240}{729}\,.\\
P(5) &= \binom{6}{5}\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^5 = \tfrac{192}{729}\,.\\
P(6) &= \binom{6}{6}\left(\tfrac{2}{3}\right)^6 = \tfrac{64}{729}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
P(\text{at least 3 red balls}) &= \tfrac{160}{729}+\tfrac{240}{729}+\tfrac{192}{729}+\tfrac{64}{729}\\
&= \tfrac{656}{729}\,.
\end{aligned}$$

Solution 2:

$$\begin{aligned}
P(0) &= \binom{6}{0}\left(\tfrac{1}{3}\right)^6 = \tfrac{1}{729}\,.\\
P(1) &= \binom{6}{1}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right) = \tfrac{12}{729}\,.\\
P(2) &= \binom{6}{2}\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^2 = \tfrac{60}{729}\,.\\
P(0)+P(1)+P(2) &= \tfrac{73}{729}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
P(\text{at least 3 red balls}) &= 1-\tfrac{73}{729}\\
&= \tfrac{656}{729}\,.
\end{aligned}$$

Depending on the particular problem, Solution 1 or Solution 2 may be the quicker method. In this particular example, there is little to choose between the two methods. If we were asked for the probability of at least five red balls, Solution 1 would be the more useful, but, if we were asked for the probability of at least two red balls, Solution 2 would be the more useful.

Example 2. In families of four children, calculate the following probabilities.

(a) Exactly three are boys.
(b) At least three are boys.
(c) At least one is a boy.

Solution:

(a) $p=\frac{1}{2}$, $q=\frac{1}{2}$, where p is the probability of a boy.

$$\begin{aligned}
P(3\text{ boys}) &= \binom{4}{3}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3\\
&= 4\times\tfrac{1}{16}\\
&= \tfrac{1}{4}\,.
\end{aligned}$$

(*b*)
$$P(4 \text{ boys}) = \binom{4}{4}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}.$$

Therefore
$$P(\text{at least 3 boys}) = \tfrac{1}{4} + \tfrac{1}{16} = \tfrac{5}{16}.$$

(c)
$$P(\text{no boys}) = \binom{4}{0}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^0 = \tfrac{1}{16}.$$

Therefore
$$P(\text{at least 1 boy}) = 1 - \tfrac{1}{16} = \tfrac{15}{16}.$$

Example 3. A student writes a test which contains ten questions. For each question, four possible answers are given and the student has to underline the only correct answer. If the student has done no studying and has to guess at all the solutions, calculate the following probabilities.

(a) He will obtain at least 50% (5 correct answers).

(b) He will obtain 100% (all answers correct).

Solution:

(a) $p = \tfrac{1}{4}, \quad q = \tfrac{3}{4}, \quad n = 10.$

$$P(5) = \binom{10}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^5 = 252\left(\frac{243}{1024}\right)\left(\frac{1}{1024}\right) = \frac{41{,}236}{1{,}048{,}576}.$$

$$P(6) = \binom{10}{6}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^6 = 210\left(\frac{81}{4^{10}}\right) = \frac{17{,}010}{1{,}048{,}576}.$$

$$P(7) = \binom{10}{7}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^7 = \frac{3{,}240}{1{,}048{,}576}.$$

$$P(8) = \binom{10}{8}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8$$

$$= \frac{405}{1{,}048{,}576}.$$

$$P(9) = \binom{10}{9}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^9$$

$$= \frac{30}{1{,}048{,}576}.$$

$$P(10) = \binom{10}{10}\left(\frac{1}{4}\right)^{10}$$

$$= \frac{1}{1{,}048{,}576}.$$

Therefore

$$P(\text{at least 5 correct}) = \frac{41{,}236 + 17{,}010 + 3{,}240 + 405 + 30 + 1}{1{,}048{,}576}$$

$$= \frac{61{,}922}{1{,}048{,}576}$$

$$\simeq .059\ .$$

(b) $$P(\text{perfect paper}) = \binom{10}{10}\left(\frac{1}{4}\right)^{10}$$

$$= \frac{1}{1{,}048{,}576}$$

$$\simeq .0000001.$$

EXERCISE 5.8

1. Four dice are thrown. Calculate each of the following probabilities.
   - (a) Two sixes show.
   - (b) Three fives show.
   - (c) At least one six shows.

2. Five coins are tossed. Calculate each of the following probabilities.
   - (a) Three show heads.
   - (b) At least three show heads.
   - (c) All five show tails.

3. In a family of five children, calculate the following probabilities.
   - (a) Three are boys.
   - (b) At least three are girls.

   (c) All five are girls.
   (d) At least one is a boy.

4. A baseball player is batting .250. Assuming that this is the probability that he will hit safely on any one time at bat, what are the following probabilities?
   (a) He will hit safely on his next twice at bat.
   (b) He will hit safely twice during his next five times at bat.

5. Records show that a certain treatment will successfully cure a particular disease in one third of the cases treated. If six patients are treated, what is the probability that at least three will be cured?

6. A pair of dice is rolled three times. Calculate each of the following probabilities.
   (a) A total of seven is rolled each time.
   (b) A total of seven is obtained once only.
   (c) A total of seven is obtained at least once.
   (d) A total of eleven is obtained at least once.

7. If the probability that a child will inherit a certain disease is 0.25, calculate the following probabilities for a family of four children.
   (a) One child will inherit the disease.
   (b) Three children will inherit the disease.
   (c) At least two of the children will be free of the disease.
   (d) All the children will inherit the disease.

8. On a test of ten questions, a student had to select the correct answer from four given possible answers. If the student knew the correct answer to three of the questions and decided to guess at the others, what is the probability that he will pass the test if five correct answers constitute a pass?

9. In a family of five children, it is known that the three eldest are boys. What is the probability that the other two are also boys?

10. If a bag contains six red balls and four black balls, and three balls are drawn without replacement, calculate the probability that all the balls drawn are red.

11. Using the data in question (10), calculate the probability that all the balls drawn are red if the balls are drawn one at a time and replaced before the next is drawn.

12. On a bridge hand of thirteen cards, the probability of obtaining no aces in any one hand is approximately 0.3. What is the probability that a player will obtain no aces in any two of three consecutive hands?

13. Calculate the probability that a bridge player will receive no honour cards (ace, king, queen, jack, or ten) on any two of three consecutive hands.

Chapter Summary

Definitions

(1) A sample space of an experiment is a set of elements such that every outcome of the experiment corresponds to exactly one element of the set.

(2) An event is a result of an experiment that may be represented by a subset of a sample space of the experiment.

(3) If A is an event corresponding to a subset of a sample space, then $\mathrm{P}(A)$, the probability of A, is the sum of the probabilities of the elements of the subset corresponding to A.

(4) Mutually-exclusive events or disjoint events are two or more events in a given sample space which have no elements in common.

$$\mathrm{P}(A \cap B) = 0\,.$$

(5) Two events, A and B, are independent if and only if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)\,.$$

(6) Two events, A and B, are dependent if and only if

$$\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \cdot \mathrm{P}(B) \quad \text{and} \quad \mathrm{P}(A \cap B) \neq 0\,.$$

Formulae

(1) For any event A,

$$0 \leq \mathrm{P}(A) \leq 1\,.$$

If $\mathrm{P}(A) = m/n$, then

$$\begin{aligned} \mathrm{P}(A') &= (n - m)/n \\ &= 1 - \mathrm{P}(A)\,. \end{aligned}$$

(2) Odds in favour of A are

$$\mathrm{P}(A) \text{ to } \mathrm{P}(A')\,.$$

If $\mathrm{P}(A) = m/n$, then

$$\mathrm{P}(A) \text{ to } \mathrm{P}(A') \text{ is } m \text{ to } n - m\,.$$

(3) $$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.$$

(4) If events A and B are mutually exclusive,

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)\,.$$

(5) For n independent events in which the probability of success in any one event is p and the probability of failure is $q = 1 - p$, the probability for exactly r successes in the n events is

$$\mathrm{P}(r) = \binom{n}{r} q^{n-r} p^{r}\,.$$

REVIEW EXERCISE 5

1. Construct sample spaces of *equally-likely outcomes* for each of the following.
   (a) Four books are to be chosen at random from six different books, A, B, C, D, E, F.
   (b) The five honour cards in spades (ace, king, queen, jack, and ten) are placed in one bundle and the five honour cards in hearts in a second bundle. One card is drawn from each bundle.

2. John, Jim, Bert, Bill, and Andy form the starting line-up for the school basketball team. The coach has obtained three tickets for a display by the Harlem Globetrotters. It is decided that the five boys will draw lots for the three tickets. Calculate the following probabilities.
   (a) John will be selected.
   (b) John and Bill will be selected.
   (c) John or Bill will be selected.
   (d) John, and Bill or Jim will be selected.
   (e) Jim and Bert, or Bill and Andy will be selected.
   (f) Andy, Jim, and Bill will be selected.

3. If two dice are thrown, state the following probabilities. (b is the number on the black die and w is the number on the white die.)
   (a) Exactly one shows a three.
   (b) Both show three.
   (c) The total is six.
   (d) The total is ten.
   (e) $b < 3$ and $w > 3$.
   (f) $b < 3$ or $w > 3$.
   (g) $b < 3$ and $w > 3$, or $b > 3$ and $w < 3$.

4. If one card is drawn from a pack of fifty-two playing cards, calculate the following probabilities.
   (a) The card is an ace.
   (b) The card is a black ace.
   (c) The card is a diamond.
   (d) The card is an ace or a king.
   (e) The card is an honour card.

5. From a group of fifteen men and ten women, a committee of three is chosen by lot. Calculate the following probabilities.
   (a) The committee contains two specified men.
   (b) The committee consists of two men and one woman.
   (c) The committee consists of three women.

6. A drawer contains eight black socks and six brown socks well-mixed. If a man opens the drawer in the dark and removes two socks, what is the probability that he has a matched par? If he removes three socks, what is the probability that he has a matched pair?

7. A bag contains seven red balls and four black balls. If three balls are drawn at random without replacement, calculate the following probabilities.
   (a) All three are red.
   (b) All three are black.
   (c) Two are red and one is black.
   (d) Two are black and one is red.

8. If four cards are drawn from a pack of fifty-two playing cards, what are the odds that none of the cards is an ace?

9. Two balls are drawn without replacement from a bag containing eight red balls and five black balls. What are the odds that both balls are red?

10. Four coins are tossed. Are the following events independent, dependent, or mutually exclusive?
    (a) First three show heads; fourth shows tails.
    (b) First shows heads; first three show heads.
    (c) First three show tails; last three show heads.
    (d) Second shows heads; fourth shows heads.

11. 5% of the population belong to blood group B and 15% of the population have the Rh^- factor in their blood. What is the probability that an individual is of blood group B and Rh^-?

12. If 40% of the population belong to blood group O, what is the probability that, of five people chosen at random, at least two belong to group O?

13. A bag contains ten red balls and five black balls. If six balls are drawn from the bag, each ball being replaced before the next is drawn, calculate the following probabilities.
    (a) Four red and two black balls are drawn.
    (b) Two red and four black balls are drawn.
    (c) At least two red balls are drawn.
    (d) At least four black balls are drawn.

14. In a batch of twenty light bulbs it is known that four are defective. In a selection of five of the bulbs, what is the probability that at least two are defective?

15. In a family of six children, calculate the following probabilities.
    (a) Two are boys and four are girls.
    (b) At least two are boys.
    (c) At least five are boys.
    (d) All are girls.

16. In a test consisting of forty-five questions, five answers are given to each question. Only one of the answers is correct and the correct answer is to be underlined. When the test is marked, it is found that 20% of those writing obtained less than eleven correct solutions. It is, therefore, decided that a score of eleven or more correct will constitute a pass. What is the probability that a student who knew nothing about the subject matter but guessed at every answer would obtain a pass? (It is suggested that logarithms be used to obtain approximations for the various probabilities involved.)

Chapter 6

POLAR CO-ORDINATES

6.1. Introduction

In this chapter we will study the polar co-ordinate system as an alternative to the Cartesian co-ordinate system for labelling points in a plane. This system will be useful in Chapter 7 in our study of complex numbers.

Select an arbitrary point O in the plane and call it the pole; select a ray with initial point at O (normally taken horizontally and to the right of O) and call it the polar axis. We will show that this point and ray form a frame of reference for a system of co-ordinates for the points in the plane. Let P be any point in the plane (Figure 6.1). Then P determines a real number r, the distance from O to P, and an angle θ (positive if measured counterclockwise from the polar axis).

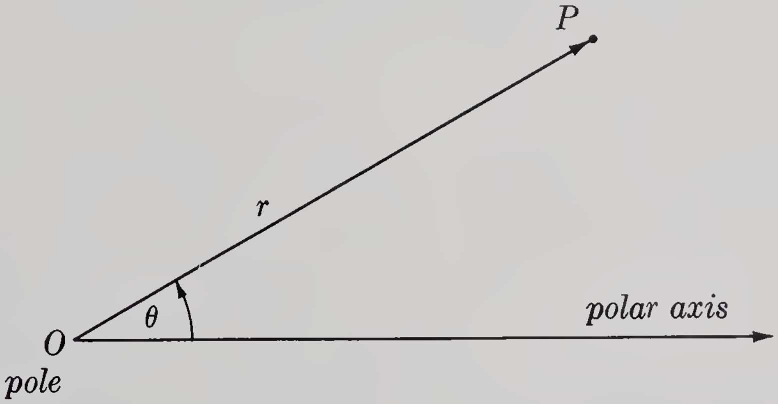


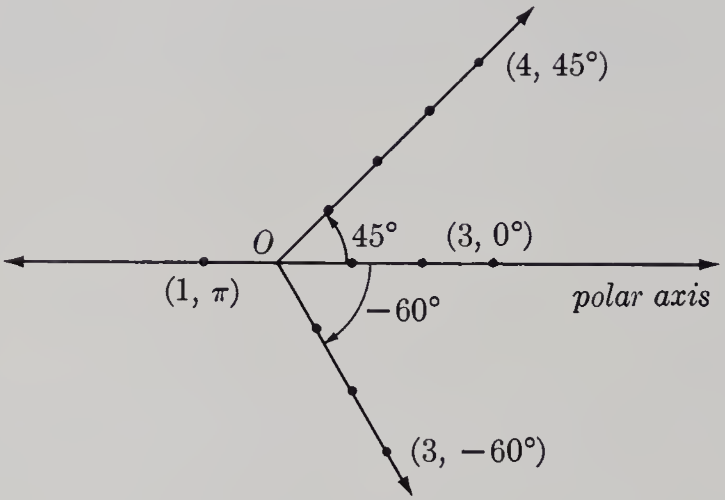
Figure 6.1

We call r the radius vector of P, and θ, the vectorial angle of P. Obviously θ is unique only up to an integral multiple of 2π radians. We call (r, θ) the polar co-ordinates of point P.

Example 1. Plot the points with the following polar co-ordinates.

(a) (3, 0°) (b) (4, 45°) (c) $(1, \pi)$ (d) (3, −60°)

Solution: In each case, in plotting (r, θ), we determine the ray with the given direction θ and then mark off r units in that direction. The plotting of polar co-ordinates is greatly simplified by the use of polar co-ordinate graph paper, as shown in the diagram.

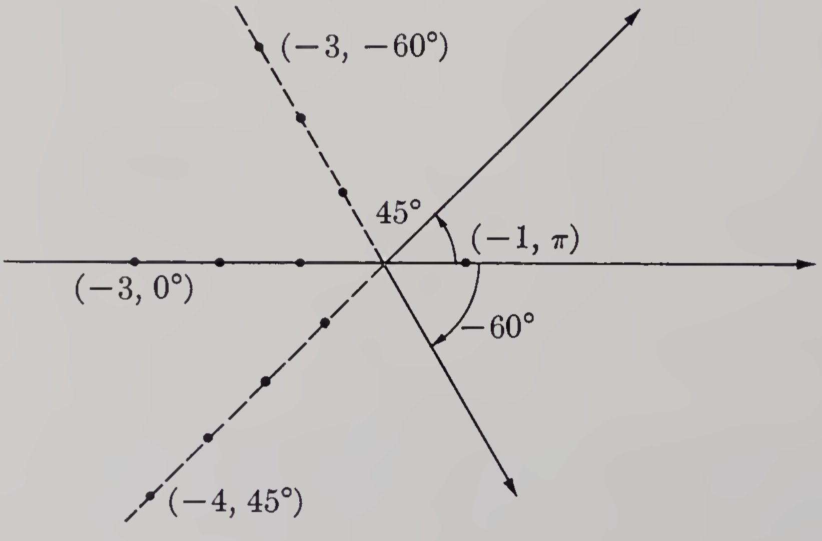


If r is a negative number, the point with polar co-ordinates (r, θ) is plotted by first determining the ray with direction θ and then marking off r units along the ray in the *opposite* direction.

Example 2. Plot the points with the following polar co-ordinates.

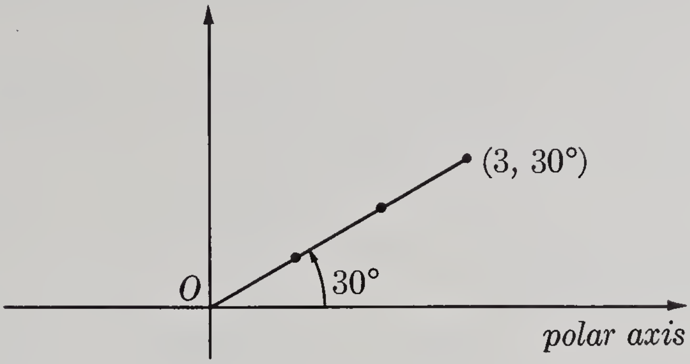
(a) (−3, 0°) (b) (−4, 45°) (c) $(-1, \pi)$ (d) (−3, −60°)

Solution:



Example 3. Give two other sets of polar co-ordinates for the point with co-ordinates (3, 30°).

Solution: (3, 390°) and (−3, 210°) are two such sets.



EXERCISE 6.1

In this exercise, use polar graph paper or a facsimile thereof for plotting points.

1. Plot the following points.
   (a) $P_1(4, 0°)$
   (b) $P_2(0, 30°)$
   (c) $P_3(5, 75°)$
   (d) $P_4(-5, 90°)$
   (e) $P_5(-3, 300°)$
   (f) $P_6\left(10\sqrt{2}, -\frac{\pi}{4}\right)$
   (g) $P_7(-12\sqrt{3}, -120°)$
   (h) $P_8(-3, 390°)$
   (i) $P_9(3, -150°)$
2. Give two additional sets of polar co-ordinates for the points in question (1).
3. Plot the following pairs of points.
   (a) $P_1(3, 30°)$ and $P_2(3, -30°)$
   (b) $Q_1(5, 60°)$ and $Q_2(5, -60°)$
   (c) $R_1(2, 135°)$ and $R_2(-2, 45°)$
   (d) $S_1(-4, 225°)$ and $S_2(4, -45°)$
4. Show that the points $P(r, \theta)$ and $Q(r, -\theta)$ or $Q(-r, 180° - \theta)$ are symmetric with respect to the polar axis.
5. Plot the following pairs of points.
   (a) $P_1(3, 30°)$ and $P_2(-3, 30°)$
   (b) $Q_1(5, 60°)$ and $Q_2(-5, 60°)$
   (c) $R_1(2, 135°)$ and $R_2(2, 315°)$
   (d) $S_1(-4, 225°)$ and $S_2(4, 225°)$
6. Show that the points $P(r, \theta)$ and $Q(-r, \theta)$ or $Q(r, 180° + \theta)$ are symmetric with respect to the pole.
7. Plot the following pairs of points.
   (a) $(4, 30°)$ and $(4, 150°)$
   (b) $(2\sqrt{2}, 45°)$ and $(2\sqrt{2}, 135°)$
   (c) $(-\sqrt{3}, 120°)$ and $(\sqrt{3}, -120°)$
   (d) $(4\sqrt{2}, 225°)$ and $(4\sqrt{2}, -45°)$

8. Show that the points (r, θ) and $(r, 180° - \theta)$ or $(-r, -\theta)$ are symmetric with respect to the line through O perpendicular to the polar axis.

9. If $0° \leq \theta < 360°$, how many sets of polar co-ordinates does every point in the plane except the pole have? How many does the pole have?

10. Plot the following points.

    (4, 0°) (4, 60°) (4, 90°) (4, 135°) (4, 180°) (4, 270°)

11. What is the locus of points for which

    (a) $r = 4$? (b) $r = a$?

12. Plot the following points.

    (1, 60°) (3, 60°) (5, 60°) (−2, 60°) (−4, 60°)

13. What is the locus of points for which

    (a) $\theta = 60°$? (b) $\theta = \theta_1$ (where θ_1 is a fixed angle)?

14. The following table of values is computed from the relation $r = 2 \cos \theta$.

θ	0°	30°	60°	90°	135°	180°	225°	270°	315°
$r = 2 \cos \theta$	2.0	1.7	1.0	0	−1.4	−2.0	−1.4	0	1.4

Plot the points (r, θ) determined by this table of values and draw a smooth curve through them. What does the curve seem to be?

15. Make a table similar to that in question (14) from the relation $r = 2 \sin \theta$, plot the corresponding points, and draw a smooth curve through them. What does the curve seem to be?

16. Repeat question (14) using the relation $r = 3 \sin \theta$.

17. What is the angle made with the polar axis by the line joining the points (3, 60°) and (4, 30°)?

18. The following table of values is computed from the relation $r = 2 \sec \theta$.

θ	0°	30°	45°	60°	85°	95°	120°	135°	150°	180°
$r = 2 \sec \theta$	2.0	2.3	2.8	4.0	22.9	−22.9	−4.0	−2.8	−2.3	−2.0

Plot the points (r, θ) determined by this table of values and draw a smooth curve through them. What does this curve seem to be?

19. Why did we not use 90° in the table in question (18)?

20. Make a table similar to that in question (18) using suitable values of θ from 180° to 360°. Draw the corresponding curve.

6.2. Relation Between Polar and Rectangular Co-ordinates

Superimpose a frame of reference for a Cartesian co-ordinate system on a frame of reference for a polar co-ordinate system by placing the origin of the former on the pole of the latter and the positive x-axis along the polar axis (Figure 6.2).

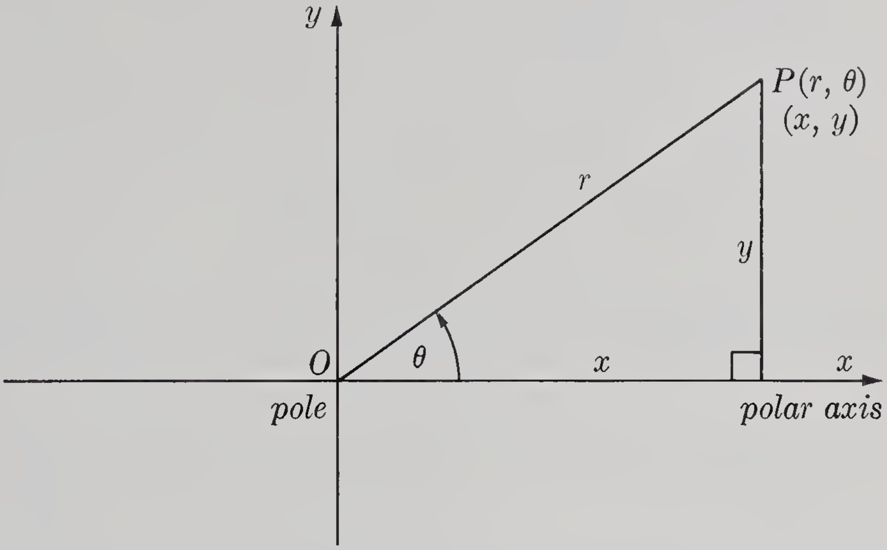


Figure 6.2

Then the point P in the plane will have rectangular co-ordinates (x, y) and polar co-ordinates (r, θ). The following relations between the rectangular and the polar co-ordinates may be read from the diagram.

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

and

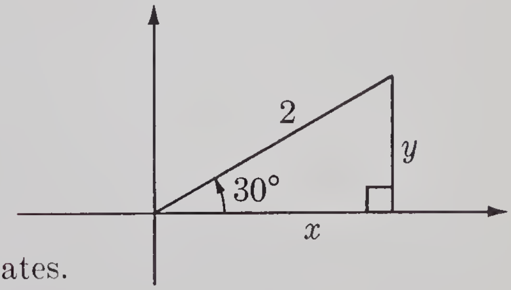
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \qquad \left(\text{that is, } \theta = \tan^{-1}\frac{y}{x}\right)$$

Example 1. Change (2, 30°) to rectangular co-ordinates.

Solution:

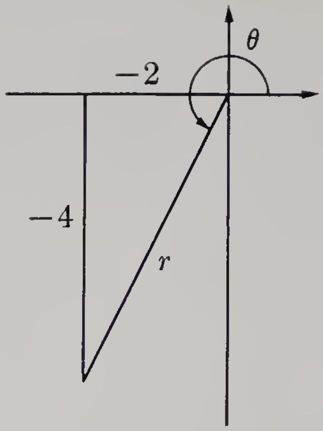
$$x = 2\cos 30° = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}\,.$$

$$y = 2\sin 30° = 2\cdot\frac{1}{2} = 1\,.$$



Therefore $(\sqrt{3}, 1)$ are the rectangular co-ordinates.

Example 2. Change $(-2, -4)$ to polar co-ordinates.



Solution:

$$r = \sqrt{(-2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$$

θ is the angle in quadrant III with

$$\tan \theta = \frac{-4}{-2} = 2\,.$$

Therefore

$$\theta = 243^\circ \qquad \text{(nearest degree).}$$

Therefore $(2\sqrt{5}, 243^\circ)$ is one set of polar co-ordinates.

Note that in changing from rectangular to polar co-ordinates, a diagram (either drawn or mental) is almost essential in order to determine the quadrant of θ.

Example 3. Change the rectangular equation $x^2 + y^2 = 2y$ to a polar equation.

Solution:

$$x^2 + y^2 = r^2 \qquad \text{and} \qquad y = r \sin \theta\,.$$

The equation $x^2 + y^2 = 2y$ becomes

$$r^2 = 2r \sin \theta\,.$$

Therefore $r = 2 \sin \theta$ is the required polar equation.

Example 4. Change the polar equation $r^2 = 4 \cos 2\theta$ to one using rectangular co-ordinates.

Solution:

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad 4 \cos 2\theta = 4(\cos^2\theta - \sin^2\theta)$$
$$= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) = \frac{4}{r^2}(x^2 - y^2)\,.$$

The equation

$$r^2 = 4 \cos 2\theta$$

becomes

$$(r^2)^2 = 4(x^2 - y^2)\,.$$

Therefore $(x^2 + y^2)^2 = 4(x^2 - y^2)$ is the required rectangular equation.

EXERCISE 6.2

1. Change the following to rectangular co-ordinates.

(a) $(4, 0°)$ (b) $(-3, 90°)$ (c) $(\sqrt{2}, 45°)$ (d) $(-6, 270°)$
(e) $(4, 30°)$ (f) $(-2, -60°)$ (g) $(5, 120°)$ (h) $(-4, 135°)$
(i) $(-3, 180°)$ (j) $(0, 45°)$ (k) $(-3, -45°)$ (l) $(-4, -90°)$

2. Change the following to polar co-ordinates $(0° \leq \theta < 360°)$.

(a) $(5, 0)$ (b) $(2, 2)$ (c) $(0, 6)$ (d) $(-4, -4)$
(e) $(-3, 0)$ (f) $(0, -5)$ (g) $(0, 0)$ (h) $(4, -4\sqrt{3})$
(i) $(-4\sqrt{3}, 4)$ (j) $(3, 4)$ (k) $(-5, 12)$ (l) $(-4, -3)$

3. Change the following to polar equations.

(a) $x = 4$. (b) $y = -3$. (c) $x^2 + y^2 = 25$.
(d) $3x - 5y = 0$. (e) $3x - 4y + 6 = 0$. (f) $xy = 4$.
(g) $y^2 - 4x = 0$. (h) $x^2 + y^2 - 2y = 0$. (i) $x^2 + y^2 = 2x + 6y$.
(j) $x^2 - y^2 = 9$. (k) $x^2 - 4y^2 = 4$. (l) $(x^2 + y^2)^2 = x^2 - y^2$.

4. Change the following to rectangular equations.

(a) $r = 6$. (b) $\theta = 30°$. (c) $r = 3 \cos \theta$.
(d) $r = \dfrac{6}{\sin \theta}$. (e) $\dfrac{r}{\cos \theta} = 4$. (f) $\dfrac{r}{\sin \theta} = 2$.
(g) $r^2 \sin \theta \cos \theta = 1$. (h) $r^2 = \dfrac{3}{\cos 2\theta}$. (i) $r^2 = \dfrac{2}{\cot \theta}$.
(j) $r = \dfrac{3}{\csc \theta}$. (k) $r = \sin 2\theta$. (l) $r = \dfrac{a}{\tan \theta}$.

5. What conic sections have the following polar equations?

(a) $r = \dfrac{2}{1 + \cos \theta}$. (b) $r = \dfrac{2}{2 + \cos \theta}$. (c) $r = \dfrac{2}{1 + 2 \cos \theta}$.

6.3. Graphing Polar Relations

We have already encountered the graphs of certain polar relations in the problems of Exercise 6.1. We found the following results.

(1) The locus of points (r, θ) satisfying the relation $r = a$ is a circle, centre the pole, and radius a. This can be verified by changing $r = a$ to Cartesian co-ordinates. Given

$$r = a,$$

then

$$\sqrt{x^2 + y^2} = a.$$

Therefore

$$x^2 + y^2 = a^2.$$

(2) The locus of points (r, θ) satisfying the relation $\theta = \theta_1$ for fixed angle θ_1 is a straight line through the origin. This again can be verified by changing to Cartesian co-ordinates. Given

$$\theta = \theta_1 ,$$

then

$$\tan \theta = \tan \theta_1 ,$$

and

$$\frac{y}{x} = m, \qquad \text{where } m = \tan \theta_1 .$$

therefore

$$y = mx .$$

(3) The locus of points (r, θ) satisfying the relation $r = a \cos \theta$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$. Given

$$r = a \cos \theta ,$$

then

$$r = a\frac{x}{r}.$$

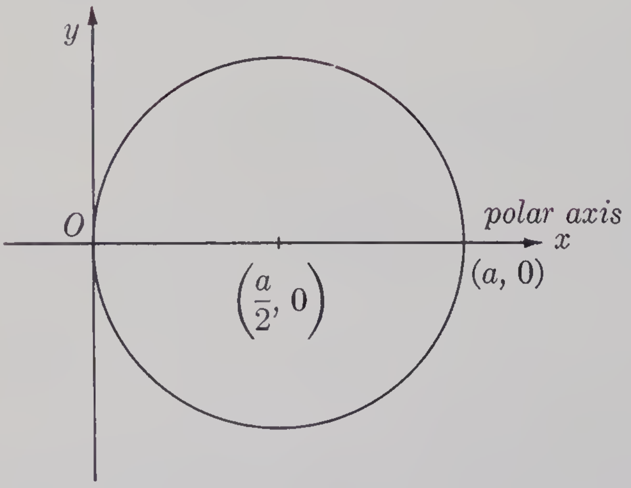
$$r^2 = ax .$$

$$x^2 + y^2 - ax = 0 .$$

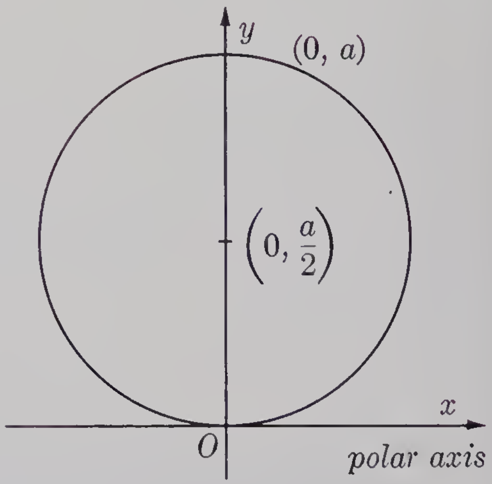
$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} .$$

Thus $r = a \cos \theta$ is the equation of a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

(4) The locus of points (r, θ) satisfying the relation $r = a \sin \theta$ is a circle with centre $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$. (It is left as an exercise for the student to convert this equation to a Cartesian equation.)



$r = a \cos \theta$



$r = a \sin \theta$

So far we have converted the given polar equation to a recognizable rectangular equation and have thus obtained the graph. If this fails, we set up a table of values of co-ordinates (r, θ) of points on the graph of the given relation, plot the corresponding points, and draw a smooth curve through them.

The following facts may prove useful in sketching graphs of polar equations.

Tangents

If θ_1 is a real root of $f(\theta) = 0$, then the curve whose equation is $r = f(\theta)$ is tangent to the line $\theta = \theta_1$ at the pole.

Example 1. In (3) above, $a \cos \theta = 0$ for $\theta = \pi/2$; the curve whose equation is $r = a \cos \theta$ is tangent to the line $\theta = \pi/2$ at the pole.

Symmetry About the Polar Axis

The points (r, θ) and $(r, -\theta)$ or $(-r, 180° - \theta)$ are symmetric with respect to the polar axis (see question (4), Exercise 6.1). Hence a polar equation which is unchanged or changed to an equivalent equation when θ is replaced by $-\theta$ or when r is replaced by $-r$ and θ by $180° - \theta$, has a graph symmetric with respect to the polar axis.

Note that the curve $r = a \cos \theta = a \cos(-\theta)$ possesses this symmetry. Also $-r = a \cos(\pi - \theta) = -a \cos \theta$ is equivalent to the equation $r = a \cos \theta$.

Example 2. Show algebraically that the curve whose equation is $r = a(1 - \cos \theta)$ is symmetric about the polar axis.

Solution:

$$a[1 - \cos(-\theta)] = a(1 - \cos \theta) .$$

Therefore the graph of $r = a(1 - \cos \theta)$ is symmetric about the polar axis.

Symmetry About the Pole

The points (r, θ) and $(-r, \theta)$ or $(r, 180° + \theta)$ are symmetric with respect to the pole (see question (6), Exercise 6.1). Hence a polar equation which is unchanged or changed to an equivalent one when r is replaced by $-r$ or θ is replaced by $180° + \theta$ has a graph which is symmetric with respect to the pole.

Example 3. Show algebraically that the graph of $r^2 = 4 \sin 2\theta$ is symmetric with respect to the pole.

Solution:

$$r^2 = (-r)^2 = 4 \sin 2\theta .$$

Hence the graph is symmetric with respect to the pole. Also

$$4 \sin 2(\theta + 180°) = 4 \sin(2\theta + 360°) = 4 \sin 2\theta ,$$

which again shows that the graph is symmetric with respect to the pole.

Symmetry About the Line $\theta = \frac{\pi}{2}$

The points (r, θ) and $(r, 180° - \theta)$ or $(-r, -\theta)$ are symmetric with respect to the line $\theta = \pi/2$ (see question (8), Exercise 6.1), as are the points (r, θ) and $(-r, -\theta)$. Hence a polar equation which is unchanged or changed into an equivalent equation when θ is replaced by $180° - \theta$, or when r is replaced by $-r$ and θ by $-\theta$, has a graph which is symmetric with respect to the line $\theta = \pi/2$.

Note that the curve $r = a \sin \theta = a \sin(\pi - \theta)$ possesses this symmetry.

Example 4. Show algebraically that the graph of $r = 4 \sin 3\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Solution:

$$\begin{aligned} 4 \sin 3(180° - \theta) &= 4 \sin (540° - 3\theta) \\ &= 4 \sin (180° - 3\theta) \\ &= 4 \sin 3\theta\,. \end{aligned}$$

Thus the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$. (See Figure 6.4.)

Again,

$$-r = 4 \sin 3(-\theta)\;;$$

that is,

$$-r = -4 \sin 3\theta\,,$$

is an equation equivalent to

$$r = 4 \sin 3\theta\,.$$

We summarize these criteria for symmetry in the following chart.

Same or equivalent equation if	implies symmetry about
θ replaced by $-\theta$ or θ replaced by $180° - \theta$ and r by $-r$	the polar axis, $\theta = 0$
θ replaced by $180° + \theta$ or r replaced by $-r$	the pole, $(0, 0)$
θ replaced by $180° - \theta$ or θ replaced by $-\theta$ and r by $-r$	the line $\theta = \frac{\pi}{2}$

These tests will be applied in the next section as an aid in plotting graphs of polar equations.

EXERCISE 6.3

1. Sketch the curves with the following polar equations. Either use a table of values to plot a number of points or change the equation to rectangular form.

(a) $r = 3$.
(b) $r = 0$.
(c) $r = -2$.
(d) $r = \cos\theta$.
(e) $r = \sin\theta$.
(f) $r = 4\cos\theta$.
(g) $r = 3\sin\theta$.
(h) $r = -2\cos\theta$.
(i) $r = -4\sin\theta$.
(j) $r = \sec\theta$.
(k) $r = \operatorname{cosec}\theta$.
(l) $r = 4\sec\theta$.
(m) $r = 4\operatorname{cosec}\theta$.
(n) $r = -\sec\theta$.
(o) $r = -3\operatorname{cosec}\theta$.

2. Find the equations of the tangent lines at the pole to the curves with the following polar equations.

(a) $r = 2\cos 2\theta$.
(b) $r = 3\cos 3\theta$.
(c) $r = 4(1 - \sin\theta)$.
(d) $r = 4(1 + \cos\theta)$.
(e) $r = 2(1 - 2\cos\theta)$.
(f) $r = 3(\cos\theta - \sin\theta)$.
(g) $r = \dfrac{2}{1 + \cos\theta}$.
(h) $r = 4\sin\theta\tan\theta$.
(i) $r = 2\cos(\theta - 30°)$.

3. Apply the tests for symmetry to the graphs with the following polar equations.

(a) $r = 4\cos 2\theta$.
(b) $r = 4\cos 3\theta$.
(c) $r = \sin 2\theta$.
(d) $r = 2\sin 4\theta$.
(e) $r^2 = \sin\theta + \cos\theta$.
(f) $r = \cos^4\theta$.
(g) $r = 4(1 - \cos\theta)$.
(h) $r = 4\tan\theta$.
(i) $r = 2 + \cos\theta$.
(j) $r = 2 + 3\cos\theta$.
(k) $r = a\theta,\ a \in Re$.
(l) $r^2 = 4\sin 2\theta$.
(m) $r^2 = 9\cos 2\theta$.
(n) $r = 2\cos(\theta - 30°)$.
(o) $r = \dfrac{2}{1 + 2\cos\theta}$.
(p) $r = \dfrac{2}{1 + \sin\theta}$.
(q) $r = 4\sin\theta\tan\theta$.
(r) $r = \cos\theta\cot\theta$.

4. State values of θ in the range $0° \leq \theta < 360°$ for which r does not have real values in the following.

(a) $r^2 = 4\sin\theta$.
(b) $r^2 = \cos 2\theta$.
(c) $r^2 + 2r = 2\cos\theta$.
(d) $r^2 = 4\sec\theta$.
(e) $r^2 = \tan\theta$.
(f) $r^3 = 8\sin\theta$.
(g) $r^2 = \dfrac{2}{1 + \sin\theta}$.
(h) $r^2 = 4\sin\theta\tan\theta$.
(i) $r^2 = \sin^3\theta$.

5. Find the points where the curves whose equations are given in question (4) intersect the polar axis.

6. Find the points common to the curves whose equations are given in the following.

(a) $r = 3(1 - \cos\theta)$ and $r = 3\cos\theta$.
(b) $r = 2\sin\theta$ and $r = 2\cos\theta$.
(c) $r = 4\sin\theta$ and $r^2 = 8\cos 2\theta$.

6.4. Further Graphing of Polar Relations

We will now consider the graphs of some additional polar relations, making use of the techniques we have discussed in the preceding section.

Example 1. Sketch the curve whose equation is

$$r = 1 + 2\cos\theta.$$

Solution:

(i) $1 + 2\cos\theta = 0$ if $\cos\theta = -\frac{1}{2}$; that is, if $\theta = 120°$ or $240°$. Thus the lines $\theta = 120°$ and $\theta = 240°$ (shown in red) are tangent to the curve at the pole.

(ii) $1 + 2\cos(-\theta) = 1 + 2\cos\theta$.
Thus the curve is symmetric with respect to the polar axis.

(iii) We know that the graph of the relation $r = 2\cos\theta$ is a circle with radius 1 and centre (1, 0) (See curve marked (i) in Figure 6.3.). We use this information to sketch the graph of $r = 1 + 2\cos\theta$, by adding 1 to each value of r corresponding to θ in $r = 2\cos\theta$. Thus, when $\theta = 0$, $r = 3$; when $\theta = 90°$, $r = 1$; the line $\theta = 120°$ is a tangent at the pole; when $\theta = 180°$, $r = -1$. Now r varies continuously as θ assumes values from 0° to 180°; this allows us to sketch the branch of the curve marked (ii) in Figure 6.3.

Now, since the curve is symmetric with respect to the polar axis, we are able to sketch in the remaining half of the curve. (This half corresponds to values of θ in the range $180° \leq \theta \leq 360°$.)

Of course, we could also have obtained the information for plotting the key points on this curve by means of a table of values:

θ	0°	60°	90°	120°	180°
$\cos\theta$	1	.5	0	$-.5$	-1
$2\cos\theta$	2	1	0	-1	-2
$r = 1 + 2\cos\theta$	3	2	1	0	-1

A similar table will give the points corresponding to values of θ between 180° and 360°.

We must emphasize that the circle whose equation is $r = 2\cos\theta$ is shown in Figure 6.3 only for comparison purposes; it is not part of the curve whose equation is $r = 1 + 2\cos\theta$.

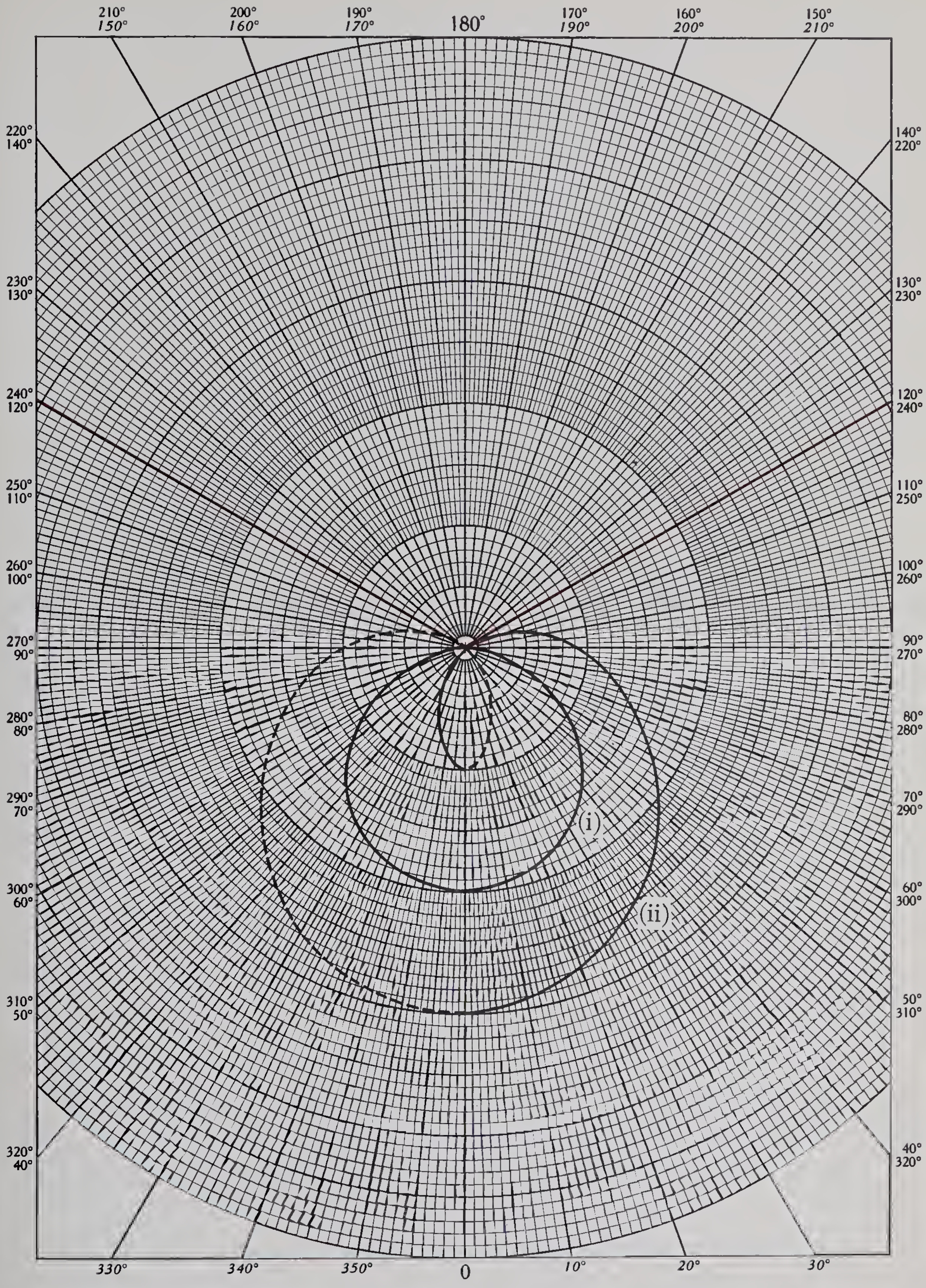


Figure 6.3. $r = 1 + 2\cos\theta$

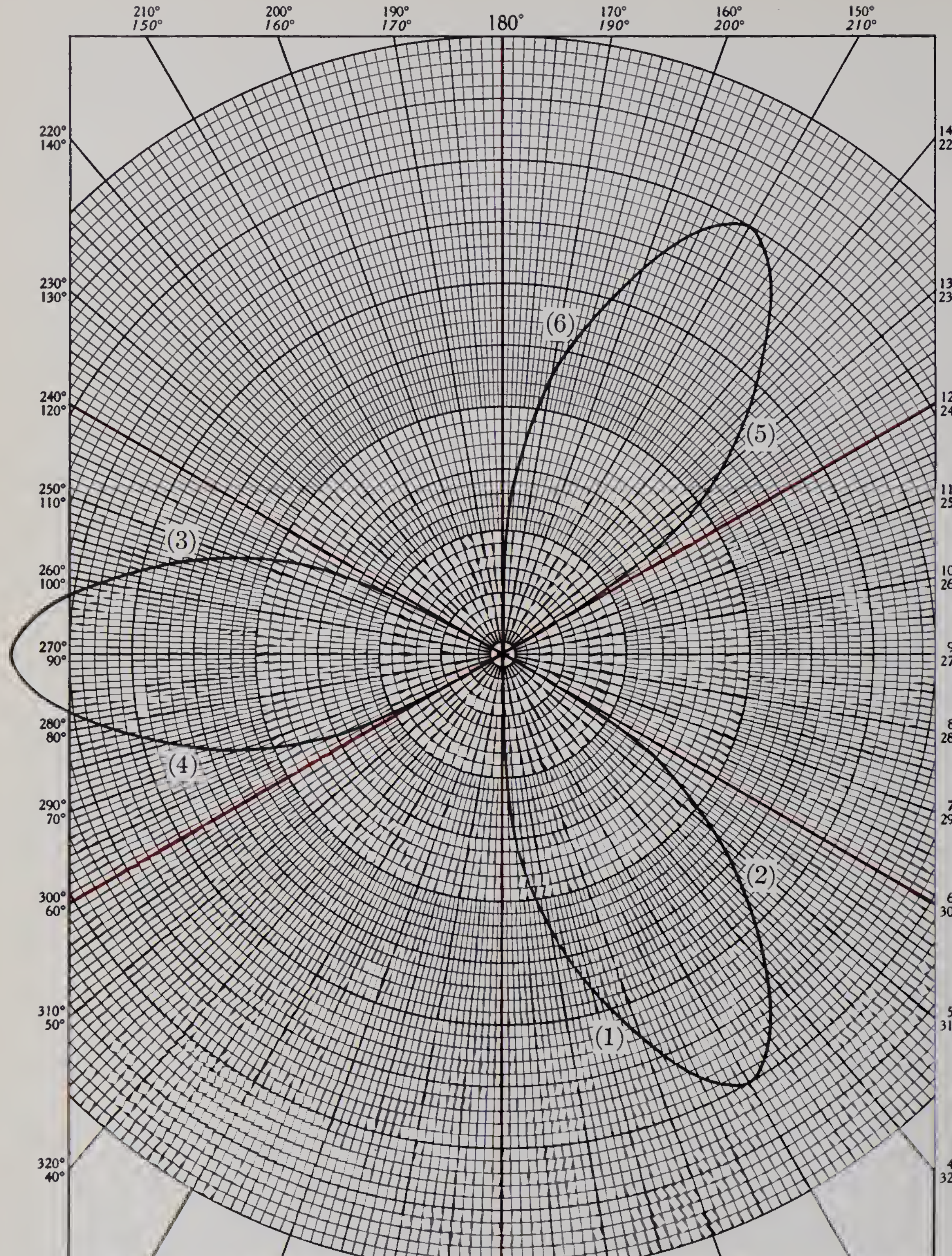


Figure 6.4. $r = 4 \sin 3\theta$ *The three-leaved rose*

Example 2. Sketch the curve whose equation is

$$r = 4 \sin 3\theta.$$

Solution:

(i) $\sin 3\theta = 0$ if $3\theta = 0°, 180°, 360°, 540°, 720°, 900°, \ldots,$
that is, if

$$\theta = 0, 60°, 120°, 180°, 240°, 300°, \ldots .$$

The lines with these equations (shown in red in Figure 6.4) are all tangent to the curve at the pole.

(ii) From Example 4, Section 6.3, we see that the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

(iii) Now r will vary continuously with θ since the sine function is a continuous function. We may set up a table of values showing the co-ordinates (r, θ) of points on the curve. In so doing we should note that we would want to assign values to θ so that we know the values of $\sin 3\theta$. For example, if $\theta = 10°$, then $\sin 3\theta = \sin 30° = 0.5$.

However, in this case it is helpful to observe that:

as 3θ varies from	0° to 90°	90° to 180°	180° to 270°	270° to 360°	360° to 450°	450° to 540°
θ varies from	0° to 30°	30° to 60°	60° to 90°	90° to 120°	120° to 150°	150° to 180°
and r varies from	0 to 4	4 to 0	0 to −4	−4 to 0	0 to 4	4 to 0
producing branch numbered	(1)	(2)	(3)	(4)	(5)	(6)

With the information from this chart, the tangent lines from (i), and the symmetry from (ii), we can sketch the curve.

From the diagram, the reader will note that the curve has symmetry about the lines whose equations are

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \frac{5\pi}{6}$$

as well as about the line whose equation is $\theta = \frac{\pi}{2}$.

These are not symmetries for which we have tests. However the reader may show that the curve whose equation is $r = f(\theta)$ is symmetric with respect to the line with equation $\theta = \frac{\pi}{6}$ if $f\left(\frac{\pi}{3} - \theta\right) = f(\theta)$. In this example, $r = 4 \sin 3\theta$ and

$$\begin{aligned} 4 \sin 3\left(\frac{\pi}{3} - \theta\right) &= 4 \sin (\pi - 3\theta) \\ &= 4 \sin 3\theta. \end{aligned}$$

Devise a similar test to show when the curve whose equation is $r = f(\theta)$ is

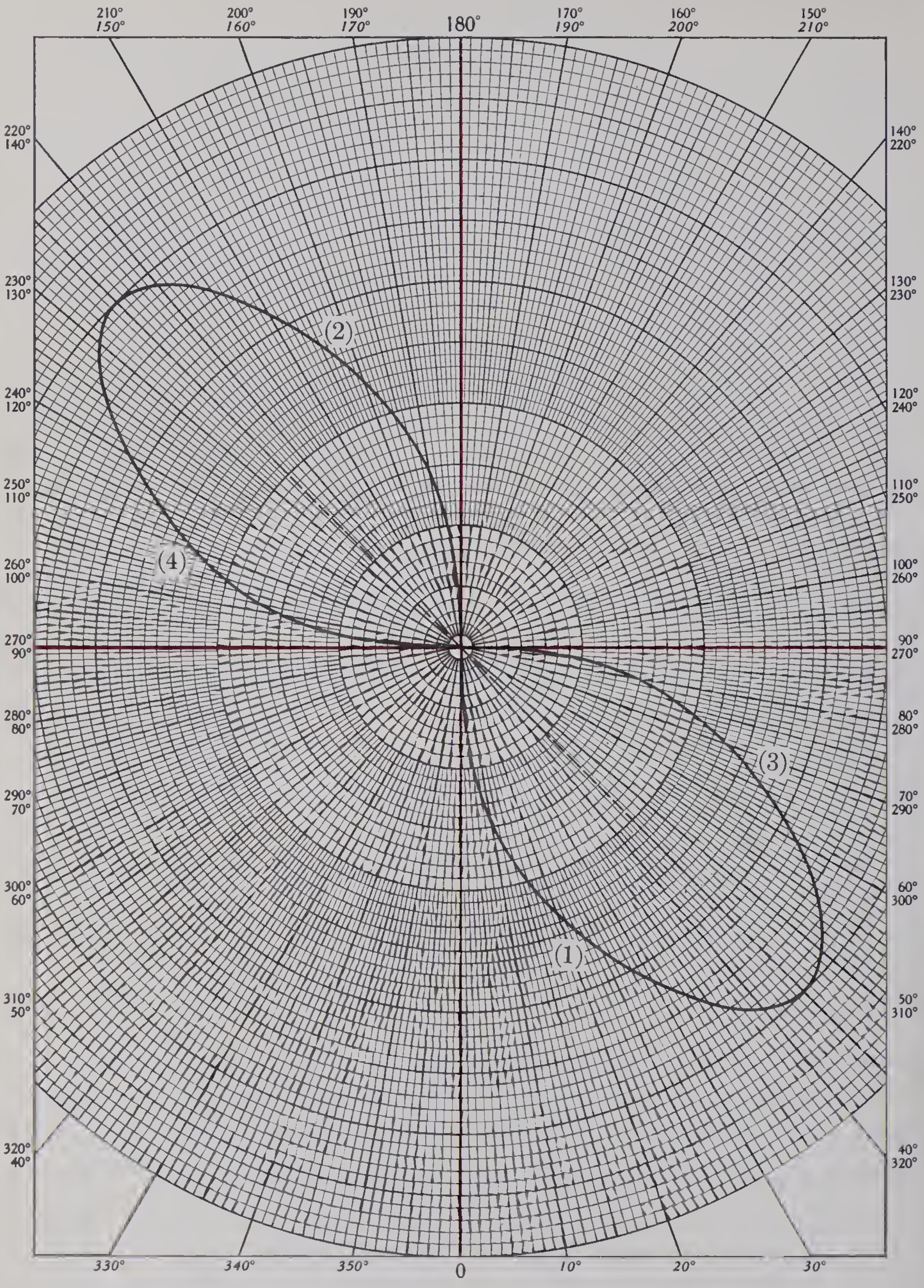


Figure 6.5. $r^2 = 4 \sin 2\theta$ *The lemniscate*

symmetric with respect to the line $\theta = \frac{5\pi}{6}$ and show algebraically that $r = 4 \sin 3\theta$ possesses this symmetry.

Example 3. Sketch the curve whose equation is

$$r^2 = 4 \sin 2\theta.$$

Solution:

(i) $4 \sin 2\theta = 0$ if

$$2\theta = 0°, 180°, 360°, 540°, \cdots,$$

that is, if

$$\theta = 0°, 90°, 180°, 270°, \cdots.$$

The lines with equations $\theta = 0°$, $\theta = 90°$, $\theta = 180°$, $\theta = 270°$ (shown in red in Figure 6.5) are tangent to the curve at the pole.

(ii) From question (31), Exercise 6.3, we see that the curve is symmetric with respect to the pole.

(iii) We note that, for real values of r, $\sin 2\theta$ must not have negative values. Hence there will be no part of the curve for values of 2θ ranging from 180° to 360° and from 540° to 720°, that is, for θ in the second and fourth quadrants.

As 2θ varies from	0 to 90°	90° to 180°
θ varies from	0 to 45°	45° to 90°
r^2 varies from	0 to 4	4 to 0
r varies from	0 to 2 (branch (1)) and 0 to −2 (branch (2))	2 to 0 (branch (3)) −2 to 0 (branch (4))

With this information, the tangent lines from (i), and the symmetry from (ii), we can sketch the curve (Figure 6.5). As θ varies from 180° to 270°, this curve is retraced.

From the diagram the reader will note symmetry about the line whose equation is $\theta = \frac{\pi}{4}$. Again the reader is asked to check that if a curve has equation $r = f(\theta)$, then it will be symmetric with respect to the line whose equation is $\theta = \frac{\pi}{4}$ if $f\left(\frac{\pi}{2} - \theta\right) = f(\theta)$. In this example, $r^2 = 4 \sin 2\theta$ and

$$\begin{aligned} 4 \sin 2\left(\frac{\pi}{2} - \theta\right) &= 4 \sin (\pi - 2\theta) \\ &= 4 \sin 2\theta. \end{aligned}$$

Similarly, this curve has symmetry with respect to the line whose equation is $\theta = \frac{3\pi}{4}$. This may be shown algebraically by checking that $f\left(\frac{3\pi}{2} - \theta\right) = f(\theta)$. In this example,

$$\begin{aligned} 4 \sin 2\left(\frac{3\pi}{2} - \theta\right) &= 4 \sin (3\pi - 2\theta) \\ &= 4 \sin (\pi - 2\theta) \\ &= 4 \sin 2\theta. \end{aligned}$$

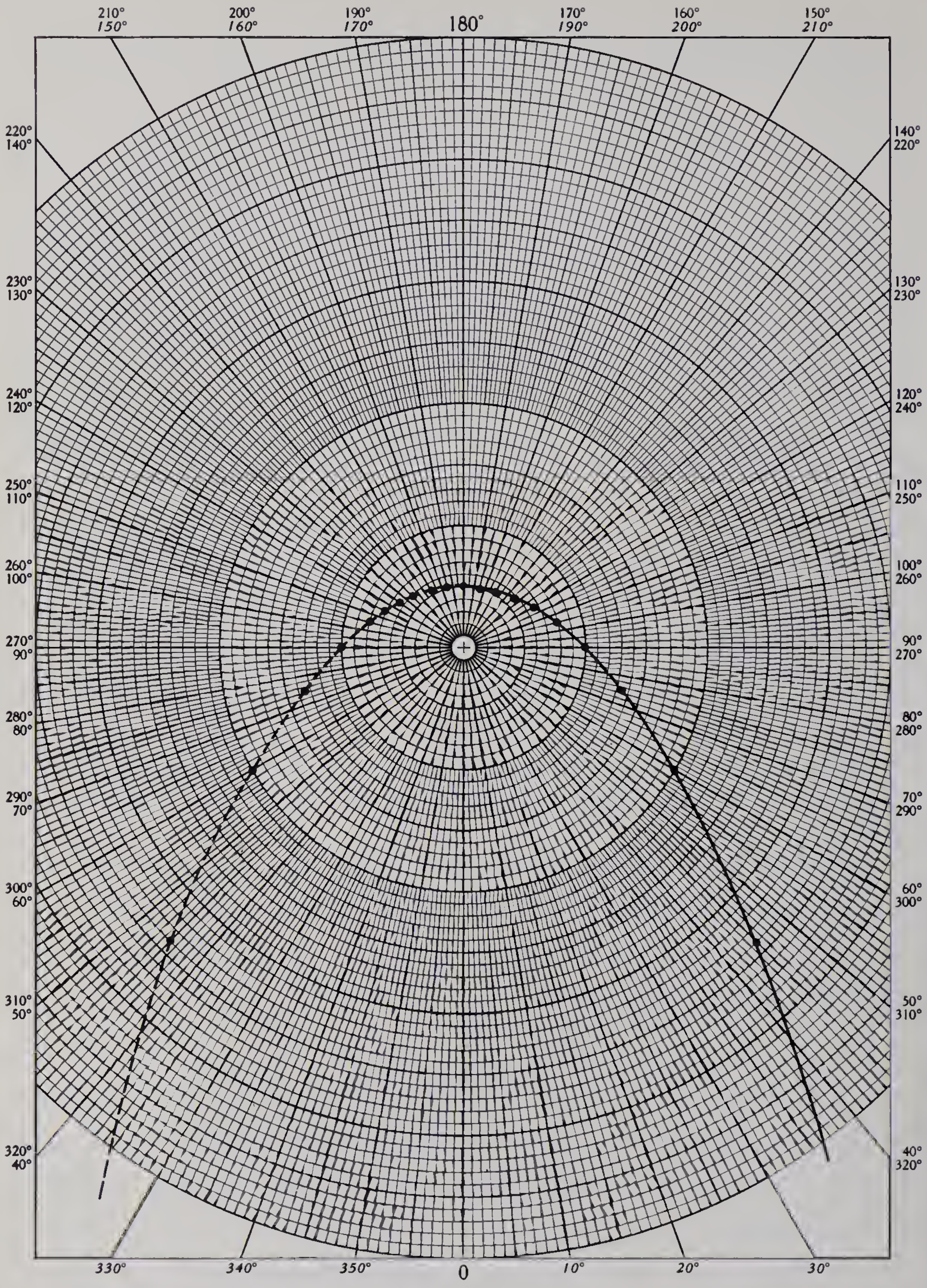


Figure 6.6. $r = \dfrac{2}{1 - \cos\theta}$

Whenever a curve possesses symmetry with respect to two lines intersecting at right angles, it possesses symmetry with respect to their point of intersection. We have already seen that this curve possesses symmetry with respect to the pole.

Example 4. Sketch the curve whose equation is

$$r = \frac{2}{1 - \cos\theta}.$$

Solution:

(i) The curve has no tangents at the pole.

(ii) The curve is symmetric about the polar axis. (Why?)

(iii) We construct the following table of values.

θ	$\cos\theta$	$1 - \cos\theta$	$r = \frac{2}{1 - \cos\theta}$
0°	1.000	0	does not exist
15°	0.966	0.034	59
30°	0.866	0.134	15
45°	0.707	0.293	6.8
60°	0.500	0.500	4
75°	0.259	0.741	2.7
90°	0	1.000	2
105°	−0.259	1.259	1.6
120°	−0.500	1.500	1.3
135°	−0.707	1.707	1.2
150°	−0.866	1.866	1.07
165°	−0.966	1.966	1.02
180°	−1.000	2.000	1

Note that as θ approaches 0°, r approaches ∞. The table shows approximations for r that can be used in our diagram.

The curve is sketched in Figure 6.6. The top branch is obtained from the points with co-ordinates (r, θ) given in the table; the bottom branch is obtained from considerations of symmetry. The curve is a parabola. This may be checked by changing to the corresponding rectangular equation.

EXERCISE 6.4

Sketch the curves with the following equations. (Refer to question (3), Exercise 6.3 for a discussion of their symmetry.)

1. $r = 4\cos 2\theta$.
2. $r = 4\cos 3\theta$.
3. $r = \sin 2\theta$.
4. $r = 2\sin 4\theta$.
5. $r^2 = \sin\theta + \cos\theta$.
6. $r = \cos^4\theta$.
7. $r = 4(1 - \cos\theta)$.
8. $r = 4\tan\theta$.
9. $r = 2 + \cos\theta$.
10. $r = 2 + 3\cos\theta$.
11. $r = a\theta,\ a \in Re$.
12. $r^2 = 4\sin 2\theta$.
13. $r^2 = 9\cos 2\theta$.
14. $r = 2\cos(\theta - 30°)$.
15. $r = \frac{2}{1 + 2\cos\theta}$.

Chapter Summary

The polar co-ordinate system · pole · polar axis · radius vector · vectorial angle

Relation between polar and rectangular co-ordinates:

$$x = r\cos\theta. \qquad r = \sqrt{x^2 + y^2}.$$

$$y = r\sin\theta. \qquad \tan\theta = \frac{y}{x}.$$

Changing from polar to rectangular equations and vice versa

Graphing polar relations

Symmetry about (a) the polar axis, (b) the pole, (c) the line $\theta = \frac{\pi}{2}$

REVIEW EXERCISE 6

1. Plot the following pairs of points.
   (a) $P_1(4, 60°)$, $Q_1(4, -60°)$ (b) $P_2(3, 120°)$, $Q_2(-3, 120°)$
   (c) $P_3(2, 45°)$, $Q_3(2, 135°)$ (d) $P_4(2, 30°)$, $Q_4(2, 210°)$
2. What symmetry is demonstrated by the pairs of points in question (1)?
3. Change the following to rectangular co-ordinates.
   (a) $(3, 45°)$ (b) $(-2, 210°)$ (c) $(-5, 1080°)$ (d) $(-3, -90°)$
4. Change the following to polar co-ordinates.
   (a) $(0, 6)$ (b) $(-2, 0)$ (c) $(-2, 2\sqrt{3})$ (d) $(-4\sqrt{3}, -4)$
5. Change to polar equations.
   (a) $x^2 + y^2 - 2x = 0.$ (b) $2x - 5y = 0.$ (c) $y^2 + 9x = 0.$
6. Change to rectangular equations.
   (a) $r\sin\theta = 2.$ (b) $r^2 = \sin 2\theta.$ (c) $r = 2\cot\theta.$
7. Sketch the curves with the following polar equations and identify the curve in each case.
   (a) $r = 4\sin\theta.$ (b) $r = 4.$ (c) $r = -4\cos\theta.$
   (d) $r = 4\csc\theta.$ (e) $r = -5\sec\theta.$ (f) $r = 2\sec\theta.$
8. Sketch the curves with the following polar equations.
   (a) $r = 4(1 - \sin\theta).$ (b) $r^2 = 9\sin 2\theta.$
   (c) $r = 1 + 2\sin\theta.$ (d) $r^2 = 4\sin\theta.$
   (e) $r = \dfrac{2}{1 - \frac{1}{2}\sin\theta}.$ (f) $r = \dfrac{2}{1 + \frac{1}{2}\cos\theta}.$
9. When is the curve whose equation is $r = f(\theta)$ symmetric about the line $\theta = \theta_1$?

Chapter 7

COMPLEX NUMBERS

7.1. Origin and Definition

Systems of numbers have been invented by mathematicians down through the centuries as the need arose. One of the basic problems confronting mathematicians of the past was the problem of solving a given algebraic equation. This problem led in turn to the invention of the negative integers, the rational numbers, the irrational numbers, and zero. However, even with the set of real numbers comprising all infinite decimal numbers, man was unable to solve all simple polynomial equations.

We have seen that the solution set of the quadratic equation

$$ax^2 + bx + c = 0$$

is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a},\quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

and that formulae exist that give the roots of the cubic and quartic equations in terms of their coefficients. (See, for example, Section 9.2 of *Senior Mathematics 2**.) However, the roots of the quadratic equation are not always real numbers; the roots of the equation

$$x^2 + x + 4 = 0$$

are

$$\frac{-1 + \sqrt{-15}}{2} \quad \text{and} \quad \frac{-1 - \sqrt{-15}}{2},$$

and these are not real numbers since there is no real number whose square is -15. Thus we have need for the invention of a new set of numbers, the set C of complex numbers.

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*Elliott, Fryer, Gardner, Hill, *Senior Mathematics 2*, Toronto, Holt, Rinehart and Winston, 1965.

We do not attempt to define "complex number" but content ourselves with the following statement. A complex number is one that can be represented by the numeral $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. Then $2 + 3i$, $4 - \frac{1}{2}i$, $\sqrt{2} + \pi i$ represent, or are numerals for, complex numbers; $3 - \sqrt{2}i$ or $3 - (\sqrt{2})i$ may be written $3 - i\sqrt{2}$ to avoid confusion with the root sign and avoid the use of parentheses.

There will be a numeral $a + bi$ corresponding to every ordered pair (a, b) of real numbers. For this reason, the ordered pair (a, b) can also be used as a numeral for the complex number it determines. Thus we may use $(2, 3)$, $(4, -\frac{1}{2})$, $(\sqrt{2}, \pi)$, and $(3, -\sqrt{2})$ as numerals for the above complex numbers.

The real number a is called the real part, or real component, or real coefficient of the complex number $a + bi$, and the real number b is called the imaginary part, or imaginary component, or imaginary coefficient of $a + bi$. We write

$$a = \operatorname{Re}\,(a + bi) \quad \text{and} \quad b = \operatorname{Im}\,(a + bi).$$

For example,

$$\operatorname{Re}\,(-3 + 2i) = -3 \quad \text{and} \quad \operatorname{Im}\,(-3 + 2i) = 2,$$

or

$$\operatorname{Re}\,(-3, 2) = -3 \quad \text{and} \quad \operatorname{Im}\,(-3, 2) = 2.$$

The name "imaginary coefficient" stems from the fact that i or $\sqrt{-1}$ was originally regarded as an "imaginary" number. Complex numbers have been in use for a long time and actually were accepted mathematically at a time when people still viewed negative numbers with suspicion. Now we realize that complex numbers are no more imaginary than any other type of number.

Complex numbers have important applications in electricity, elasticity, hydrodynamics, nucleonics, etc., branches of physical science and engineering in which the real number system is not sufficient to describe the physical processes.

If the complex number $x + yi$ is thought of essentially as the ordered pair (x, y) of real numbers, there is an obvious geometric representation of the complex number by the point in the Cartesian plane with co-ordinates (x, y). When we so represent a complex number, however, we do not use the names "x- and y-axes" for the perpendicular lines forming the frame of reference, and "Cartesian plane" for the plane so determined. Rather, we call these lines the real axis and the imaginary axis, and we call the resulting plane the complex plane or Argand plane (named after Jean Robert Argand who, in 1806, gave this geometrical representation of a complex number and used it to show that every algebraic equation has a root).

Example 1. Represent geometrically the complex numbers

$$1 + i, \quad -5 + 4i, \quad -3 - 2i, \quad 4 - 3i, \quad 6 + 0i, \quad 0 - 2i.$$

Solution: These complex numbers are represented, respectively, by the points $(1, 1)$, $(-5, 4)$, $(-3, -2)$, $(4, -3)$, $(6, 0)$, and $(0, -2)$ in the complex or Argand plane (Figure 7.1).

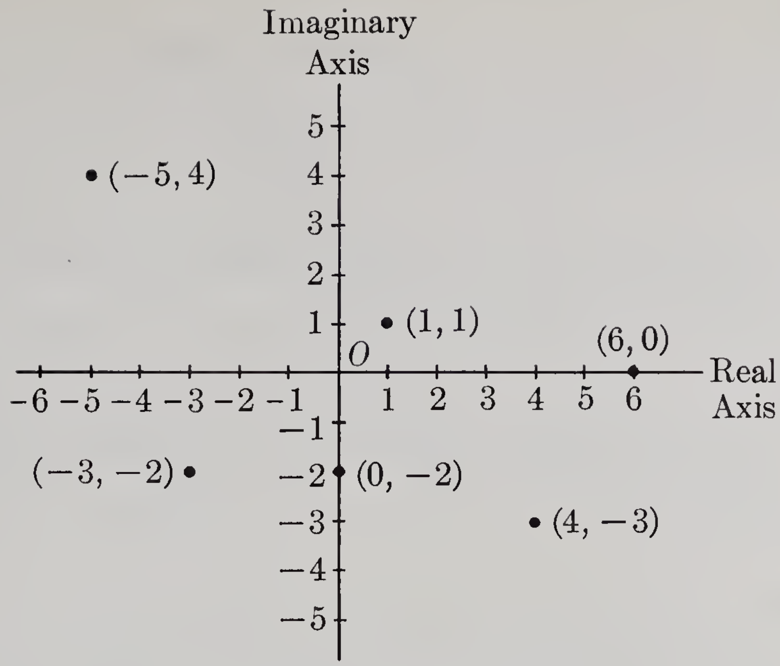


Figure 7.1. Argand Plane

Complex numbers of the form $a + 0i$ usually are written simply a. We will see later that there is a close connection between the subset of complex numbers of the form $a + 0i$ and the set of real numbers.

The complex number $x + yi$ also may be represented by the vector joining the origin to the point in the Argand plane with co-ordinates (x, y). Figure 7.2 shows the complex numbers of Example 1 represented as vectors.

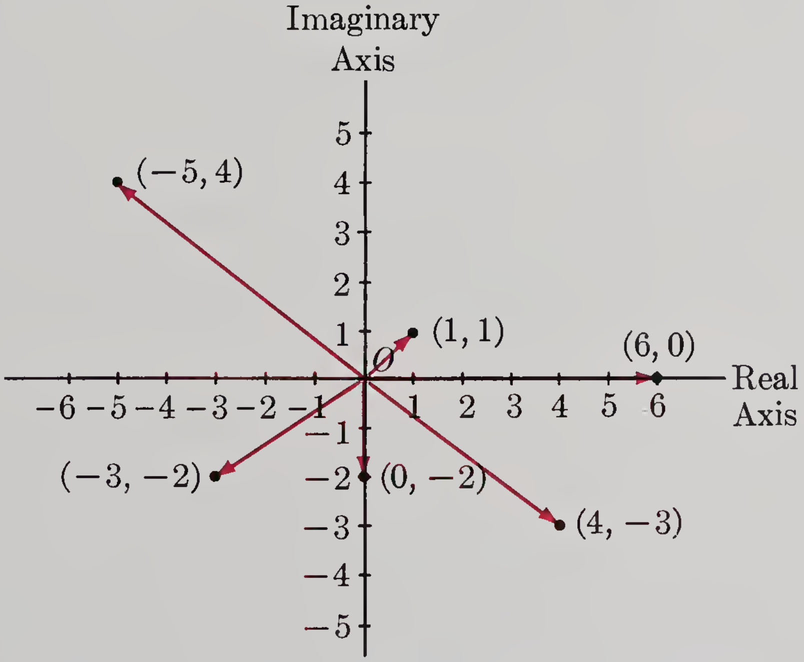


Figure 7.2. Complex numbers represented by vectors

The representation of a complex number as a vector allows us to associate an absolute value with a complex number; this is the absolute value or length of the vector representing the complex number.

DEFINITION. The absolute value or modulus or length $|x + yi|$ of the complex number $x + yi$ is defined to be

$$|x + yi| = \sqrt{x^2 + y^2}.$$

Example 2. Find the absolute values of the complex numbers of Example 1.

Solution:

$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}.$

$|-3 - 2i| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}.$

$|6 + 0i| = \sqrt{6^2 + 0^2} = 6.$

$|-5 + 4i| = \sqrt{5^2 + 4^2} = \sqrt{41}.$

$|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5.$

$|0 - 2i| = \sqrt{0^2 + (-2)^2} = 2.$

EXERCISE 7.1

1. Give (i) the real coefficient and (ii) the imaginary coefficient of the following.
   (a) $6 - i\sqrt{2}$ (b) $\frac{1}{2} + \frac{1}{2}i$ (c) $-\pi + 6i$ (d) $2x + 3yi$
   (e) $(4, 3)$ (f) $(0, -5)$ (g) $(\frac{1}{2}, -\sqrt{2})$ (h) $(5\sqrt{2} + 3, 0)$
2. Express the following complex numbers in the form $a + bi$.
   (a) $(3, -2)$ (b) $(0, 6)$ (c) $(2, 3 + \sqrt{2})$ (d) (x, y)
3. Express the following complex numbers in the form (a, b).
   (a) $5 - \frac{3}{2}i$ (b) $-\sqrt{2} + 3i$ (c) $\frac{2}{3} - \pi i$ (d) $p + qi$
4. Represent the following complex numbers by points in the Argand plane.
   (a) $6 + 4i$ (b) $-3 + 2i$ (c) $\frac{1}{2} + \frac{3}{2}i$ (d) $-4 - 2i$
   (e) $0 + 3i$ (f) $-3 + 0i$ (g) $\sqrt{2} + i\sqrt{2}$ (h) $\pi + 4i$
5. Give the absolute values of the complex numbers in question (4).
6. Represent the following complex numbers by vectors.
   (a) $3 + 4i$ (b) $5 - 2i$ (c) $-3 - 4i$ (d) $-2 + 6i$
   (e) $4 + 0i$ (f) $0 - 3i$ (g) $-3 + i\sqrt{2}$ (h) $-5 + 0i$
7. Give the absolute values of the complex numbers in question (6).
8. What complex number does the origin of the Argand plane represent?
9. What is the locus of points representing complex numbers of the form $0 + bi$?
10. What is the locus of points representing complex numbers of the form $a + 0i$?
11. What is the locus of points representing complex numbers whose real coefficients are equal to (a) their imaginary coefficients? (b) the negative of their imaginary coefficients?
12. Sketch the region in the Argand plane which represents complex numbers with absolute values less than or equal to unity.

7.2. Addition of Complex Numbers

In order to study the arithmetic of complex numbers, we must first know when two complex numbers are equal.

DEFINITION. $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Thus two complex numbers are equal if and only if their real components are equal and their imaginary components are equal.

Example 1. Are the following complex numbers equal?

(a) $2 + 3i$ and $\pi + i\sqrt{2}$ (b) $x + yi$ and $2 - \frac{1}{2}i$

Solution:

(a) $2 + 3i \neq \pi + i\sqrt{2}$ since $\operatorname{Re}(2 + 3i) = 2$ and $\operatorname{Re}(\pi + i\sqrt{2}) = \pi$ and $\pi \neq 2$; the imaginary parts are also unequal in this case.

(b) $x + yi = 2 - \frac{1}{2}i$ if and only if $x = 2$ and $y = -\frac{1}{2}$.

Example 2. For what values of m and n will

$$(m + n) + (2m - n)i = 3 + 9i?$$

Solution: We have equality if and only if

$$m + n = 3$$

and

$$2m - n = 9.$$

Therefore $m = 4$, $n = -1$ are the required values.

If we assume that the expressions $a + bi$ and $c + di$ behave like linear polynomials in the symbol i, with the additional property that $i^2 = -1$, we may justify the result of the above definition as follows:

If

$$a + bi = c + di,$$

then

$$a - c = i(d - b),$$

and

$$(a - c)^2 = -(d - b)^2;$$

that is,

$$(a - c)^2 + (d - b)^2 = 0.$$

But $(a - c)^2$ and $(d - b)^2$ are both nonnegative real numbers and their sum can equal zero only if each is zero; therefore

$$a - c = b - d = 0 \quad \text{if} \quad a + bi = c + di.$$

Conversely, if $a = c$ and $b = d$, then certainly $a + bi = c + di$.

DEFINITION. Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

This definition tells us that we may add complex numbers exactly as we add polynomials; for example,

$$(2 + 3x) + (4 + 7x) = 6 + 10x\,,$$

and similarly,

$$(2 + 3i) + (4 + 7i) = 6 + 10i\,.$$

The definition also shows us immediately that the set of complex numbers is closed under addition.

According to our definition,

$$(a + bi) + (0 + 0i) = a + bi\,,$$

so that the complex number $0 + 0i$ is the identity or zero for addition of complex numbers. If there is no confusion, we will write simply 0 for this complex number.

We note that, by this definition of addition,

$$(7 - 3i) + (-7 + 3i) = 0 + 0i\,,$$

so that $(-7 + 3i)$ is the negative of the complex number $7 - 3i$; that is,

$$-7 + 3i = -(7 - 3i)\,.$$

In general,

$$(a + bi) + (-a - bi) = (a - a) + (b - b)i = 0$$

and so

$$-(a + bi) = -a - bi \text{ is the additive inverse of } a + bi.$$

We may now define subtraction of complex numbers as follows.

DEFINITION. Subtraction:

$$(a + bi) - (c + di) = (a + bi) + (-c - di).$$

Thus to subtract a complex number, we add its additive inverse.

Example 3.

(a) Express $-(-3 + 2i)$ as a complex number in the form $a + bi$.

(b) Express $(5 + 4i) - (-3 + 2i)$ as a complex number in the form $a + bi$.

Solution:

(a) $$-(-3 + 2i) = 3 - 2i\,.$$

(b)
$$\begin{aligned}(5+4i)-(-3+2i) &= (5+4i)+[-(-3+2i)]\\ &= (5+4i)+(3-2i)\\ &= 8+2i\,.\end{aligned}$$

The fact that addition of complex numbers is carried out in terms of addition of real numbers (the real and imaginary components of the complex numbers), shows us that addition of complex numbers is commutative and associative. For example,

$$\begin{aligned}(3+2i)+(-7+4i) &= -4+6i\\ &= (-7+4i)+(3+2i)\,.\end{aligned}$$

In general

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\ &= (c+a)+(d+b)i\\ &= (c+di)+(a+bi)\,.\end{aligned}$$

Thus addition of complex numbers is commutative.

Addition of complex numbers is associative; for example,

$$\begin{aligned}(4+i)+[(-3+2i)+(6-4i)] &= (4+i)+(3-2i)\\ &= 7-i\,;\\ [(4+i)+(-3+2i)]+(6-4i) &= (1+3i)+(6-4i)\\ &= 7-i\,.\end{aligned}$$

EXERCISE 7.2

1. Find the values of x and y so that the complex numbers in the following pairs are equal.

   (a) $6x+3yi\,,\ 4+2i$ (b) $x-3yi\,,\ -\pi-i\sqrt{2}$
   (c) $x-(4-y)i\,,\ -\frac{1}{2}+\frac{3}{2}i$ (d) $x-(x+y)i\,,\ 3+0i$
   (e) $(x+2y)+3xi\,,\ -6+0i$ (f) $(x+2y)+3xi\,,\ 0-6i$
   (g) $(x+y)+(2x-3y)i\,,\ 3-2i$ (h) $(4x-y)+(\frac{1}{2}x+y)i\,,\ -3-2i$

2. Add the following pairs of complex numbers.

   (a) $4+3i\,,\ 7+2i$ (b) $6-3i\,,\ -2+4i$ (c) $-\frac{1}{2}+3i\,,\ -\frac{3}{2}-2i$
   (d) $6-4i\,,\ 6+4i$ (e) $-\frac{1}{2}-\frac{3}{2}i\,,\ \frac{5}{2}+i$ (f) $-4+\pi i\,,\ 4+\pi i$
   (g) $\sqrt{2}+i\,,\ 3+2i$ (h) $6\sqrt{2}+i\,,\ -3+i\sqrt{2}$ (i) $\pi+2i\,,\ \sqrt{3}+i\sqrt{2}$

3. State the additive inverses of the following in the form $a+bi$.

   (a) $5-2i$ (b) $-3\sqrt{2}+\frac{1}{2}i$ (c) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

4. In the following, subtract the second complex number from the first.
   (a) $6 + i,\ -4 + i$ (b) $3 - 2i,\ -2 - 5i$ (c) $-6 - 2i,\ -5 - 4i$
   (d) $4 - \frac{1}{2}i,\ \frac{3}{2} + \frac{1}{2}i$ (e) $\frac{1}{2} + 3i,\ -2 - \frac{1}{2}i$ (f) $6 + 4i,\ -6 + 4i$
   (g) $\sqrt{2} + 3i,\ 5\sqrt{2} - i$ (h) $3\sqrt{2} + 4i,\ \pi + i\sqrt{2}$ (i) $\dfrac{1}{\sqrt{3}} + i,\ 2 - i\sqrt{2}$

5. Verify that addition of complex numbers is associative using the following triples of complex numbers.
   (a) $1 - i,\ 3 - 4i,\ 5 + 2i$ (b) $-6 + 2i,\ -4 - 3i,\ 5 - 6i$

6. Prove that the set of complex numbers is closed under subtraction.

7. Prove that addition of complex numbers is associative.

8. Solve the following equations for z.
   (a) $(2 + 3i) + z = -4 + i.$ (b) $(-1 + 2i) + z = 5 - \frac{1}{2}i.$
   (c) $(\sqrt{2} + i) - z = 4 + i\sqrt{2}.$ (d) $6 - i = z + (5 - \sqrt{2})i.$

9. Represent the complex numbers $z_1 = 5 + i$, $z_2 = 2 + 3i$, and their sum $z_1 + z_2$ geometrically.

10. Prove that the origin and the points representing z_1, z_2, and $z_1 + z_2$ in question (9) are the vertices of a parallelogram.

11. Prove that the result in question (10) is true for every pair of complex numbers z_1 and z_2.

12. Interpret the result in question (11) using the vector representation for complex numbers.

7.3. Multiplication of Complex Numbers

DEFINITION. Multiplication:

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i.$$

This rather complicated looking definition tells us that we may multiply complex numbers in exactly the way that we multiply polynomials, bearing in mind that i^2 may be replaced by -1. For example,

$$\begin{aligned}(2 + 3x) \times (4 + x) &= 8 + 2x + 12x + 3x^2\\ &= 8 + 14x + 3x^2\,.\end{aligned}$$

Similarly,

$$\begin{aligned}(2 + 3i) \times (4 + i) &= 8 + 2i + 12i + 3i^2\\ &= 8 + 14i + 3(-1)\\ &= 5 + 14i\,.\end{aligned}$$

$$\begin{aligned}(a+bi)\times(c+di) &= ac+adi+bci+bdi^2\\ &= ac+(ad+bc)i+bd(-1)\\ &= (ac-bd)+(ad+bc)i\,.\end{aligned}$$

The definition shows us immediately that the set of complex numbers is closed under multiplication.

The following examples indicate that multiplication of complex numbers is commutative and associative.

Example 1. Evaluate

(a) $(2-3i)\times(5+i)$; (b) $(5+i)\times(2-3i)$.

Solution:

(a)
$$\begin{aligned}(2-3i)\times(5+i) &= 10+2i-15i-3i^2\\ &= 13-13i\,.\end{aligned}$$

(b)
$$\begin{aligned}(5+i)\times(2-3i) &= 10-15i+2i-3i^2\\ &= 13-13i\,.\end{aligned}$$

Therefore

$$(2-3i)\times(5+i)=(5+i)\times(2-3i)\,.$$

Example 2. Evaluate

(a) $(1+2i)\times[(6-i)\times(2+3i)]$, (b) $[(1+2i)\times(6-i)]\times(2+3i)$.

Solution:

(a)
$$\begin{aligned}&(1+2i)\times[(6-i)\times(2+3i)]\\ &= (1+2i)\times[12-2i+18i-3i^2]\\ &= (1+2i)\times(15+16i)\\ &= 15+16i+30i+32i^2\\ &= -17+46i\,.\end{aligned}$$

(b)
$$\begin{aligned}&[(1+2i)\times(6-i)]\times(2+3i)\\ &= [6-i+12i-2i^2]\times(2+3i)\\ &= (8+11i)\times(2+3i)\\ &= 16+24i+22i+33i^2\\ &= -17+46i\,.\end{aligned}$$

Therefore

$$(1+2i)\times[(6-i)\times(2+3i)]=[(1+2i)\times(6-i)]\times(2+3i)\,.$$

Note that

$$(a+bi)\times(1+0i)=a+bi\,,$$

so that $1+0i$ is the multiplicative identity or unit for the multiplication of complex numbers. When there can be no confusion, we will write 1 instead of $1+0i$.

Example 3. If $z_1 = 2 - 5i$, $z_2 = -3 + 4i$, and $z_3 = -1 - 6i$, find

(a) $z_1(z_2 + z_3)$, (b) $z_1z_2 + z_1z_3$.

Solution:

(a)
$$\begin{aligned} z_2 + z_3 &= (-3 + 4i) + (-1 - 6i) \\ &= -4 - 2i\,. \end{aligned}$$

Therefore
$$\begin{aligned} z_1(z_2 + z_3) &= (2 - 5i) \times (-4 - 2i) \\ &= -18 + 16i\,. \end{aligned}$$

(b)
$$\begin{aligned} z_1z_2 &= (2 - 5i) \times (-3 + 4i) \\ &= 14 + 23i\,. \\ z_1z_3 &= (2 - 5i) \times (-1 - 6i) \\ &= -32 - 7i\,. \end{aligned}$$

Therefore
$$\begin{aligned} z_1z_2 + z_1z_3 &= (14 + 23i) + (-32 - 7i) \\ &= -18 + 16i\,. \end{aligned}$$

We conclude that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ for this particular triple of complex numbers z_1, z_2, and z_3.

If you examine closely the behaviour of complex numbers of the form $a + 0i$ under addition and multiplication, you will see that they behave in exactly the same way as the real numbers a.

complex numbers	*real numbers*
$(6 + 0i) + (2 + 0i) = 8 + 0i$	$6 + 2 = 8$
$(6 + 0i) \times (2 + 0i) = 12 + 0i$	$6 \times 2 = 12$

Because of this, mathematicians agree to identify the complex number $a + 0i$ with the real number a and even to write $a + 0i = a$ (as long as there is no possibility of confusion).

EXERCISE 7.3

1. Evaluate the following.

(a) i^2 (b) i^5 (c) i^{16} (d) i^{99} (e) i^{999}
(f) $-3i \times 2i^5$ (g) $\frac{1}{2}i^3 \times 4i$ (h) $-6i \times 4i^{19}$ (i) $i^3 \times i^{97}$ (j) $6i^{10} \times 4i^4$

2. Multiply the following pairs of complex numbers.

(a) $4 + 3i\,, 7 + 2i$ (b) $6 - 3i\,, -2 + 4i$ (c) $-\frac{1}{2} + 3i\,, -\frac{3}{2} - 2i$
(d) $6 - 4i\,, 6 + 4i$ (e) $-\frac{1}{2} - \frac{3}{2}i\,, \frac{5}{2} + i$ (f) $-4 + \pi i\,, 4 + \pi i$

(g) $\sqrt{2} + i,\ 3 + 2i$ (h) $6\sqrt{2} + i,\ -3 + i\sqrt{2}$

(i) $4\sqrt{2} - 3i,\ \sqrt{2} + 2i$ (j) $-5 + i\sqrt{2},\ -6 - 3i\sqrt{2}$

3. Use the following pairs of complex numbers to verify by example that multiplication of complex numbers is commutative.

   (a) $6 + i,\ -4 + i$ (b) $3 - 2i,\ -2 - 5i$ (c) $\frac{1}{2} + 3i,\ -2 - \frac{1}{2}i$

4. Prove that multiplication of complex numbers is commutative.

5. Use the following triples of complex numbers to verify by example that multiplication of complex numbers is associative and distributive over addition.

   (a) $3 - i,\ 5 + i,\ 2 - 3i$ (b) $6 + 2i,\ 5 - i\sqrt{2},\ 5 + i\sqrt{2}$

6. (a) Verify that

   $$(1 + 2i)[(3 - 2i) + (-5 - 4i)] = (1 + 2i)(3 - 2i) + (1 + 2i)(-5 - 4i).$$

   (b) What law is exemplified in (a)?

7. Find uv, $u^2 + v^2$, and $u^3 - v^3$, if $u = 2 - i$ and $v = -1 + 2i$.

8. Find w^3 if $w = 1 + i\sqrt{3}$.

9. If $u^3 = w^3$, does it necessarily follow that $u = w$?

10. If $u^2 + v^2 = 0$, does it necessarily follow that $u = v = 0$?

11. If $w = \cos\theta + i\sin\theta$, find w^2 and show that w^2 is equal to $\cos 2\theta + i\sin 2\theta$.

12. Write the multiplication table for the set of four numbers $\{1, i, -1, -i\}$, thereby showing that this set is closed under multiplication.

13. Prove that the product of two nonzero complex numbers is always nonzero.

14. Let $z = 4 + 3i$ and plot z, iz, i^2z, and i^3z on the same diagram. Note that i may be considered as a rotation operator under multiplication; describe the rotation effect of i on z.

15. If $z = 1 - 2i\sqrt{2}$, plot z, z^2, and z^3 on the same diagram.

16. Express the following complex numbers in the form $a + bi$.

   (a) $i^{17} + i^{28} + i^{43}$ (b) $(-1 + 2i)^4$

   (c) $\dfrac{(1 - i)^3}{\sqrt{2}}$ (d) $(4 - i)(5 + 2i)(3 - i)$

17. Prove that multiplication of complex numbers is associative and distributive over addition.

7.4. Complex Conjugates

DEFINITION. Conjugate complex numbers, or complex conjugates, are pairs of complex numbers differing only in the sign of their imaginary coefficients.

For example, $6 - 3i$ and $6 + 3i$, $-4 + 0i$ and $-4 - 0i$, $0 - 6i$ and $0 + 6i$ are pairs of conjugate complex numbers or complex conjugates.

Example 1. State the complex conjugates of the complex numbers

(a) $4 - 3i$, (b) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$,

and in each case find the product of the given complex number and its conjugate.

Solution:

(a) The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i) \times (4 + 3i) = 16 - 9i^2 = 25\,.$$

(b) The complex conjugate of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

$$\begin{aligned}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \times \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) &= \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 i^2 \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1\,.\end{aligned}$$

If we use the single letter z to represent the complex number $x + yi$, then we use the symbol $\bar{z}$ to represent the complex conjugate $x - yi$. We have the following theorems.

Theorem 1. $z \times \bar{z}$ is a real number.

Proof:

$$\begin{aligned} z \times \bar{z} = (x + yi) \times (x - yi) &= (x^2 + y^2) + 0i \\ &= x^2 + y^2\,. \end{aligned}$$

(Recall that we have agreed to identify the real number a with the complex number $a + 0i$.)

Theorem 2. $z \times \bar{z} = |z|^2$.

Proof: From the preceding theorem, if $z = x + yi$, then

$$z \times \bar{z} = x^2 + y^2\,.$$

Also

$$|z| = \sqrt{x^2 + y^2}\,.$$

Therefore

$$z \times \bar{z} = |z|^2\,.$$

Example 2. Given $z_1 = 3 - 2i$ and $z_2 = -2 + 5i$, compare

$$\overline{z_1 + z_2} \quad \text{and} \quad \overline{z_1} + \overline{z_2}\,.$$

Solution:

$$\begin{aligned} z_1 + z_2 &= (3 - 2i) + (-2 + 5i) \\ &= 1 + 3i\,. \end{aligned}$$

Hence

$$\overline{z_1 + z_2} = 1 - 3i\,.$$

$$\overline{z_1} = 3 + 2i \quad \text{and} \quad \overline{z_2} = -2 - 5i\,.$$

Hence

$$\begin{aligned} \overline{z_1} + \overline{z_2} &= (3 + 2i) + (-2 - 5i) \\ &= 1 - 3i\,. \end{aligned}$$

Therefore

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

for these particular complex numbers z_1 and z_2.

The result in this example is proved in general in the following theorem.

Theorem 3. If z_1 and z_2 represent complex numbers, then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\,;$$

that is, the conjugate of the sum of two complex numbers is equal to the sum of their conjugates.

Proof: Let

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2\,.$$

Then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)\,.$$

Also

$$\overline{z_1} = x_1 - iy_1 \quad \text{and} \quad \overline{z_2} = x_2 - iy_2\,,$$

so that

$$\begin{aligned} \overline{z_1} + \overline{z_2} &= (x_1 - iy_1) + (x_2 - iy_2)\,. \\ &= (x_1 + x_2) - i(y_1 + y_2)\,. \end{aligned}$$

Therefore

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\,.$$

Example 3. Given $z_1 = 2 - 5i$ and $z_2 = -3 + 2i$, compare

$$\overline{z_1 \times z_2} \quad \text{and} \quad \overline{z_1} \times \overline{z_2}\,.$$

Solution:

$$\begin{aligned} z_1 \times z_2 &= (2 - 5i) \times (-3 + 2i) \\ &= 4 + 19i\,. \end{aligned}$$

Hence

$$\overline{z_1 \times z_2} = 4 - 19i\,.$$

$$\overline{z_1} = 2 + 5i \quad \text{and} \quad \overline{z_2} = -3 - 2i$$

Hence

$$\begin{aligned} \overline{z_1} \times \overline{z_2} &= (2 + 5i) \times (-3 - 2i) \\ &= 4 - 19i\,. \end{aligned}$$

Therefore

$$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$$

for these particular complex numbers z_1 and z_2.

The result in this example is stated in general in the following theorem.

Theorem 4. If z_1 and z_2 represent complex numbers, then

$$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}\,;$$

that is, the conjugate of the product of two complex numbers is equal to the product of their conjugates.

Proof: The proof is left as an exercise. (See question (9), Exercise 7.4.)

Example 4. Given $z = -4 - 9i$, compare $\overline{\overline{z}}$ and z.

Solution:

$$\begin{aligned} z &= -4 - 9i\,. \\ \overline{z} &= -4 + 9i\,. \\ \overline{\overline{z}} &= -4 - 9i\,. \end{aligned}$$

Therefore

$$\overline{\overline{z}} = z$$

for this particular complex number z.

The result in this example is stated in general in the following theorem; its proof is left as an exercise. (See question (10), Exercise 7.4.)

Theorem 5. If z represents a complex number, then

$$\overline{\overline{z}} = z\,;$$

that is, the conjugate of the conjugate of a complex number is equal to that complex number.

EXERCISE 7.4

1. State the conjugates of the following complex numbers.
   (a) $4 - 3i$ (b) $-5 - 4i$ (c) $4 + 0i$
   (d) $-3 - 3i$ (e) $0 - 5i$ (f) $8 + 5i$
2. Find $z + z$ and $z - z$ for each complex number in question (1).
3. Are any complex numbers equal to their own conjugates; if so, which ones?
4. Multiply the following complex numbers by their conjugates.
   (a) $6 - i$ (b) $-2 - 4i$ (c) $\frac{1}{2} + \frac{1}{2}i$ (d) $0 + 4i$
   (e) $2 - i\sqrt{3}$ (f) $4\sqrt{3} + 2i$ (g) $\frac{1}{2} + 0i$ (h) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
5. What is meant by the statements "The product of a complex number and its conjugate is a *real* number." and "The sum of a complex number and its conjugate is a *real* number."?
6. What is the connection between the geometrical representations of a complex number and its conjugate?
7. If $z_1 = 1 + i$, $z_2 = 4 - 3i$, and $z_3 = -1 - 2i$, find
   (a) $\overline{z_1 + z_2 + z_3}$ and $\overline{z_1} + \overline{z_2} + \overline{z_3}$,
   (b) $\overline{z_1 \times z_2 \times z_3}$ and $\overline{z_1} \times \overline{z_2} \times \overline{z_3}$,
   (c) $\overline{z_1 \times (z_2 + z_3)}$ and $\overline{z_1} \times \overline{z_2} + \overline{z_1} \times \overline{z_3}$.
8. Repeat question (7) if $z_1 = 5 - 2i$, $z_2 = -1 - i$, and $z_3 = 0 - 4i$.
9. Prove that $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$ for complex numbers z_1 and z_2.
10. Prove that $z_1 = z_1$ for any complex number z_1.
11. Prove that $\overline{z_1 + z_2 + z_3} = \overline{z_1} + \overline{z_2} + \overline{z_3}$ for complex numbers z_1, z_2, and z_3.
12. Prove that $\overline{kz} = k\overline{z}$ if k is a real number and z is a complex number.
13. Prove that $\overline{z_1 \times z_2 \times z_3} = \overline{z_1} \times \overline{z_2} \times \overline{z_3}$ for complex numbers z_1, z_2, and z_3.

7.5. Division of Complex Numbers

In this section we consider the problem of dividing complex numbers.

Example 1. Express $\dfrac{2 - 3i}{4 + i}$ in the form $a + bi$.

Solution: We use the fact that the product of a complex number and its conjugate is essentially a real number. Since $4 - i$ is the conjugate of the denominator and $(4 - i)(4 + i) = 17$, we multiply the given expression by $\dfrac{4 - i}{4 - i}$ (which is equal to 1, the identity element for multiplication).

Then

$$\begin{aligned}\frac{2-3i}{4+i} &= \frac{2-3i}{4+i} \times \frac{4-i}{4-i}\\ &= \frac{5-14i}{17}\\ &= \frac{5}{17} - \frac{14}{17}i\,.\end{aligned}$$

Note that our operation here is similar to "rationalizing the denominator" (actually we are "realizing" the denominator). We may check that

$$\frac{2-3i}{4+i} = \frac{5}{17} - \frac{14}{17}i\,,$$

by showing that

$$2 - 3i = (4+i)\left(\frac{5}{17} - \frac{14}{17}i\right).$$

Division of a complex number by a nonzero complex number is always possible as in the example. The set of complex numbers is closed under division by nonzero complex numbers.

If $(a+bi)(c+di) = 1 + 0i = 1$, then $c+di$ is the multiplicative inverse of $a+bi$; by the definition of division as the inverse operation to multiplication, we have

$$c + di = \frac{1}{a+bi}\,.$$

Example 2. Find the multiplicative inverse of $4 - 3i$.

Solution: We are asked to find $\dfrac{1}{4-3i}$.

Now

$$\begin{aligned}\frac{1}{4-3i} &= \frac{1}{4-3i} \times \frac{4+3i}{4+3i}\\ &= \frac{4+3i}{16+9}\\ &= \frac{4}{25} + \frac{3}{25}i\,.\end{aligned}$$

Check:

$$\begin{aligned}(4-3i)\left(\frac{4}{25} + \frac{3}{25}i\right) &= \frac{16}{25} - \frac{12}{25}i + \frac{12}{25}i - \frac{9}{25}i^2\\ &= 1 + 0i\\ &= 1\,.\end{aligned}$$

Every nonzero complex number will have a multiplicative inverse. (See question (4), Exercise 7.5.) As usual, the zero complex number $0 + 0i$ does not have a multiplicative inverse.

We are now in a position to review the properties of addition and multiplication as defined for complex numbers and to show that all the field postulates hold for the set of complex numbers with these operations.

(i) The set C of complex numbers is closed under the binary operations of addition and multiplication.

(ii) Addition of complex numbers is associative and commutative, $0 + 0i$ is the additive identity, and $-a - bi$ is the additive inverse of the complex number $a + bi$.

(iii) Multiplication of complex numbers is associative and commutative, $1 + 0i$ is the multiplicative identity, and every nonzero complex number has a multiplicative inverse.

(iv) Finally, multiplication of complex numbers is distributive over addition. Thus the complex numbers form a field under our definition of addition and multiplication.

There is one significant way in which the set of complex numbers differs from the set of real numbers. The set C of complex numbers is *not* an ordered set; that is, given two complex numbers z_1 and z_2, we can attach no significant meaning to either $z_1 > z_2$ or $z_1 < z_2$.

EXERCISE 7.5

1. Divide the first complex number in each of the following pairs by the second.

   (a) $4 + 3i,\ 7 + 2i$ (b) $6 - 3i,\ -2 + 4i$ (c) $3 + 2i,\ -4 - 2i$
   (d) $-\frac{1}{2} + 3i,\ -\frac{3}{2} + 2i$ (e) $6 - 4i,\ 6 + 4i$ (f) $-\frac{1}{2} - \frac{3}{2}i,\ \frac{5}{2} + i$
   (g) $-4 + \pi i,\ 4 + \pi i$ (h) $\sqrt{2} + i,\ 3 + 2i$ (i) $6\sqrt{2} + i,\ -3 + i\sqrt{2}$

2. Evaluate the following quotients in the form $a + bi$; multiply the resulting numbers to check. (The product should be 1.)

   (a) $\dfrac{6 - 3i}{1 + 2i}, \dfrac{1 + 2i}{6 - 3i}$ (b) $\dfrac{2 + i}{4 - 3i}, \dfrac{4 - 3i}{2 + i}$ (c) $\dfrac{1 + i\sqrt{2}}{4 + i\sqrt{3}}, \dfrac{4 + i\sqrt{3}}{1 + i\sqrt{2}}$

3. Find the multiplicative inverses of the following complex numbers. Check each result.

   (a) $4 + i$ (b) $3 - 2i$ (c) $6 + 3i$ (d) $-6 + 2i$
   (e) $\frac{1}{2} - \frac{1}{2}i$ (f) $\sqrt{3} + i$ (g) $2 - i\sqrt{2}$ (h) $\sqrt{3} + i\sqrt{2}$

4. Find the complex number which is the multiplicative inverse of the nonzero complex number $a + bi$.

5. Solve the following equations for z.

   (a) $(1 + i)z = 3 - 2i$ (b) $(-2 + 3i)z = 2 - 5i$
   (c) $(6 - 2i)z = -\frac{1}{2} + 3i$ (d) $(4 - 3i)z = 2$

6. Prove that the product of two complex numbers is zero if and only if at least one of the two complex numbers is zero.

7.6. Solution of Quadratic Equations

If a is a positive real number, the symbol $\sqrt{\ }$ in $\sqrt{a}$ can be considered as indicating a certain arithmetical operation; if a is a negative real number, it can no longer be so considered. However, just as $\sqrt{2}$ may be defined as a real number which is such that $\sqrt{2} \times \sqrt{2} = 2$, and by convention we take it to be the positive such number, so we may define $\sqrt{-2}$ to be a number which is such that $\sqrt{-2} \times \sqrt{-2} = -2$.

DEFINITION. If $a \in Re^+$, then $(\sqrt{-a})^2 = -a$.

Then, by the definition,

$$\sqrt{-1} \times \sqrt{-1} = -1\,;$$

that is, $\sqrt{-1} = i$. Further, if $a > 0$,

$$(i\sqrt{a}) \times (i\sqrt{a}) = i^2a = (-1)a\,;$$

that is,

$$(i\sqrt{a})^2 = -a\,.$$

Then, by convention,

$$i\sqrt{a} = \sqrt{-a} \text{ if } a > 0\,.$$

Example 1. Evaluate the following.

(a) $\sqrt{16} \times \sqrt{9}$ (b) $\sqrt{-16} \times \sqrt{9}$

(c) $\sqrt{16} \times \sqrt{-9}$ (d) $\sqrt{-16} \times \sqrt{-9}$

Solution:

(a) $\sqrt{16} \times \sqrt{9} = 4 \times 3 = 12$
$= \sqrt{9 \times 16}.$

(b) $\sqrt{-16} \times \sqrt{9} = 4i \times 3 = 12i$
$= \sqrt{-144} = \sqrt{-16 \times 9}.$

(c) $\sqrt{16} \times \sqrt{-9} = 4 \times 3i = 12i$
$= \sqrt{-144} = \sqrt{16 \times -9}.$

(d) $\sqrt{-16} \times \sqrt{-9} = 4i \times 3i = 12i^2 = -12$
$\neq \sqrt{(-16) \times (-9)} = \sqrt{144} = 12.$

The results of Example 1 illustrate the validity of the following theorem.

Theorem. If $a, b \in Re$, then

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

if and only if at least one of a or b is nonnegative.

Proof: The proof is required in question (11), Exercise 7.6.

With complex numbers at our disposal, we are now in a position to solve the general quadratic equation $ax^2 + bx + c = 0$. The solution set is

$$\left\{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right\}.$$

If

$$b^2 - 4ac \geq 0,$$

then the roots are real; if

$$b^2 - 4ac < 0,$$

then

$$b^2 - 4ac = -(4ac - b^2) \quad \text{and} \quad 4ac - b^2 > 0\,.$$

Hence, in this latter case,

$$\begin{aligned} \sqrt{b^2 - 4ac} &= \sqrt{-(4ac - b^2)} \\ &= \sqrt{-1}\sqrt{4ac - b^2} \\ &= i\sqrt{4ac - b^2}\,, \end{aligned}$$

and the roots are the conjugate complex numbers

$$\frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} \quad \text{and} \quad \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}\,.$$

Example 2. Solve the equation $x^2 + x + 4 = 0$.

Solution: The solution set is $\left\{\frac{-1 \pm \sqrt{-15}}{2}\right\}$; that is,

$$\left\{-\frac{1}{2} + \frac{i\sqrt{15}}{2},\ -\frac{1}{2} - \frac{i\sqrt{15}}{2}\right\}.$$

With the invention of complex numbers, mathematicians reached the end of their quest as far as the solution of equations was concerned. No matter what polynomial equation is written, even with complex coefficients, the roots will be complex numbers. We cannot write a polynomial equation that does not have such a solution; there is never a need for a different type of number in such a situation.

The famous *Fundamental Theorem of Algebra* states the following.

The polynomial equation of degree n,

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0\,,$$

in which the coefficients are complex numbers, always has at least one complex root. (The word *complex* includes, of course, real, rational, integral.)

With this theorem, which is not easily proved, we may readily prove by induction that such an equation has exactly n complex roots.

We have just considered the formula that gives the roots of the quadratic equation $ax^2 + bx + c = 0$ in terms of the coefficients a, b, and c; similar formulae exist for the cubic equation $ax^3 + bx^2 + cx + d = 0$ and the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$. Such formulae do not (and *cannot*) exist for equations of higher degree; however these equations still have roots. Various numerical methods exist to find the roots of such equations to any required degree of accuracy.

Example 3. Solve the equation $2iz^2 - (4 + 3i)z + (3 - i) = 0$.

Solution: By the quadratic formula, the solution set is

$$\left\{\frac{4 + 3i \pm \sqrt{(4 + 3i)^2 - 4(2i)(3 - i)}}{4i}\right\}.$$

Now

$$\begin{aligned}(4 + 3i)^2 - 4(2i)(3 - i) &= 16 + 24i - 9 - 24i - 8\\ &= -1,\end{aligned}$$

and so the roots are

$$\begin{aligned}\frac{4 + 4i}{4i} &= \frac{1}{i} + 1 &\quad \text{and} \quad \frac{4 + 2i}{4i} &= \frac{1}{i} + \frac{1}{2}\\ &= 1 - i. & &= \frac{1}{2} - i.\end{aligned}$$

The solution set is $\{1 - i, \frac{1}{2} - i\}$.

EXERCISE 7.6

1. Express the following as complex numbers in the form $a + bi$.
   (a) $\sqrt{-25}$ (b) $\sqrt{16} - \sqrt{-9} - \sqrt{-64}$ (c) $\sqrt{-4}\,\sqrt{25}$
   (d) $\sqrt{81}\,\sqrt{-16}$ (e) $\sqrt{-9}\,\sqrt{-9}$ (f) $\sqrt{-100}\,\sqrt{36}$
   (g) $\sqrt{-1}\,\sqrt{-8}$ (h) $\sqrt{-65}\,\sqrt{-13}$ (i) $\sqrt{-42}\,\sqrt{-14}$

2. Find the complex roots of the following equations.
   (a) $6x^2 - x + 2 = 0$. (b) $3x^2 - \sqrt{2}x + 1 = 0$.
   (c) $\sqrt{2}x^2 - 5x + 6\sqrt{2} = 0$. (d) $\sqrt{3}x^2 - x + \frac{4}{\sqrt{3}} = 0$.
   (e) $z^2 - iz + 4 = 0$. (f) $3z^2 + 2iz - 1 = 0$.
   (g) $iz^2 - 6z + 3i = 0$. (h) $2iz^2 + z - 3i = 0$.

3. Solve the equation of Example 3 of this section by first multiplying both members by i.

4. Find the complex roots of the following equations $ax^2 + bx + c = 0$; verify that the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$.

   (a) $x^2 + 5x + 8 = 0$. (b) $2x^2 - 4x + 3 = 0$.

   (c) $6x^2 - 5x + 2 = 0$. (d) $3x^2 + x + 2 = 0$.

5. Show that the sum of the roots of the equation $ax^2 + bx + c = 0$ is always $-\frac{b}{a}$ if the coefficients a, b, and c are real numbers.

6. In question (5), show that the product of the roots is always $\frac{c}{a}$.

7. What are the roots of the equation

$$[x - (1 + i)][x - (2 - i)] = 0\,?$$

8. Write the equation in question (7) in the form $ax^2 + bx + c = 0$. Does the sum of the roots equal $-\frac{b}{a}$? Does the product of the roots equal $\frac{c}{a}$? Explain.

9. Show that if the roots of a quadratic equation with real coefficients are complex numbers, they are conjugate complex numbers. Why is this not violated in question (7)?

10. Form quadratic equations with the following pairs of roots.

    (a) $2 \pm i$ (b) $-3 \pm 2i$ (c) $0 \pm i$

    (d) $-\sqrt{2} \pm 3i$ (e) $6 \pm i\sqrt{3}$ (f) $\frac{1}{2} \pm \frac{1}{2}i\sqrt{2}$

11. Prove the theorem of this section: If $a, b \in Re$, then

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

if and only if at least one of a or b is nonnegative.

7.7. The Polar Form of a Complex Number

The complex number $x + yi$ is represented geometrically by the point P in the Argand plane with co-ordinates (x, y) or, equivalently, by the vector joining the origin to this point (Figure 7.3). In this way we have a one-to-one correspondence between the set of complex numbers and the set of points in the Argand plane (or the set of vectors in the Argand plane with initial point at the origin).

We have already associated a magnitude or length or absolute value with the complex number $z = x + yi$, namely, the nonnegative length of the vector representing the complex number,

$$|x + yi| = \sqrt{x^2 + y^2}.$$

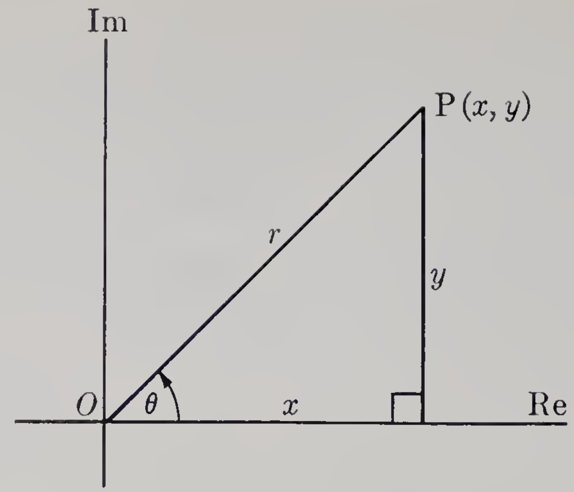


Figure 7.3

Usually we use the letter r to represent the length of the vector associated with the complex number z;

$$r = |z| = \sqrt{x^2 + y^2}\,.$$

Now the vector OP determines an angle θ (a positive number of radians is measured counterclockwise from the real or horizontal axis). This angle is called the argument of the complex number z;

$$\theta = \operatorname{Arg} z\,.$$

The term "angle of z" is also used for θ. Note that a complex number z does not determine a unique angle θ, since obviously

$$\operatorname{Arg} z = \theta + 2k\pi\,, \qquad k \in I.$$

Thus each complex number z will determine an infinite number of ordered pairs (r, θ) consisting of a nonnegative real number and an angle. However such an ordered pair will determine a unique point in the Argand plane and thus determine a unique complex number z. There is a many-to-one correspondence between the set of ordered pairs (r, θ) and the set of complex numbers.

From Figure 7.3 we see that

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{y}{r} = \sin\theta\,;$$

that is,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,.$$

We may then write

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta)\,. \end{aligned}$$

$x + yi$ is called the Cartesian or rectangular form of the complex number and $r(\cos \theta + i \sin \theta)$ is known as the polar form of the complex number z (so called because r and θ are polar co-ordinates for the corresponding point in the plane; polar co-ordinates were discussed in Chapter 6). This form is often abbreviated as

$$z = r(\cos\theta + i \sin\theta) = r \operatorname{cis}\theta,$$

read "$r \cos\theta$ plus $i \sin\theta$." Of course, since

$$\cos\theta = \cos(\theta + 2k\pi) \quad \text{and} \quad \sin\theta = \sin(\theta + 2k\pi),$$

the polar form of a complex number is not unique.

$$z = r \operatorname{cis}\theta = r \operatorname{cis}(\theta + 2k\pi).$$

This is a very important fact in later work with complex numbers.

There is no difficulty in changing from the polar form

$$z = r(\cos\theta + i\sin\theta)$$

to the rectangular form

$$x + yi;$$

we merely evaluate $\cos\theta$ and $\sin\theta$ (possibly using tables) and multiply by r.

Example 1. Find the rectangular form of the complex number

$$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Solution: Since

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

therefore

$$z = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2 + 2i.$$

Example 2. Find the rectangular form of the complex number

$$z = 3 \operatorname{cis} 127°.$$

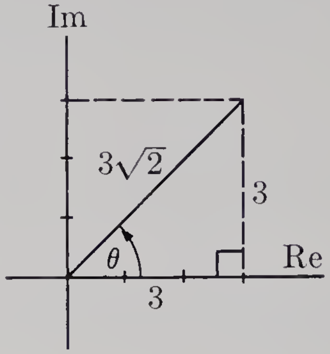
Solution: From tables, $\cos 127° = -0.6018$ and $\sin 127° = 0.7986$. Therefore

$$\begin{aligned} z &= 3(-0.6018 + 0.7986i) \\ &= -1.8054 + 2.3958i. \end{aligned}$$

In changing from rectangular to polar form, a diagram such as that in Figure 7.3 is useful. We plot the complex number in rectangular form and indicate r and θ on the diagram.

Example 3. Find the polar form of the complex number $z = 3 + 3i$.

Solution:



$$r = \sqrt{3^2 + 3^2} = 3\sqrt{2}\,.$$

$$\tan\theta = \frac{3}{3} = 1\,,$$

and

$$\theta \text{ is in quadrant I.}$$

Hence

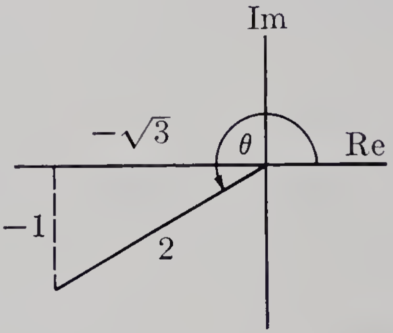
$$\theta = \frac{\pi}{4}\,,$$

and therefore

$$z = 3\sqrt{2}\,\operatorname{cis}\left(\frac{\pi}{4} + 2k\pi\right).$$

Example 4. Find the polar form of the complex number $z = -\sqrt{3} - i$.

Solution:



$$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2\,.$$

$$\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

and

$$\theta \text{ is in quadrant III}\,.$$

Hence

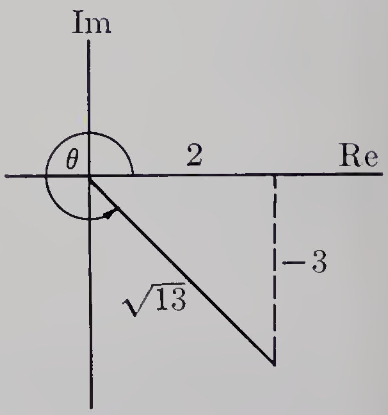
$$\theta = 210^\circ\,,$$

and therefore

$$z = 2\,\operatorname{cis}\,(210 + 360k)^\circ\,.$$

Example 5. Find the polar form of the complex number $z = 2 - 3i$.

Solution:



$$r = \sqrt{2^2 + (-3)^2} = \sqrt{13}\,.$$

$$\tan\theta = \frac{-3}{2} = -1.5$$

and

$$\theta \text{ is in quadrant IV}\,.$$

Hence

$$\theta = 304^\circ, \qquad \text{(to nearest degree from tables),}$$

and therefore

$$z = \sqrt{13}\,\operatorname{cis}\,(304 + 360k)^\circ\,.$$

EXERCISE 7.7

1. Plot each of the following complex numbers and write them in rectangular form.

   (a) $4(\cos 0° + i \sin 0°)$ (b) $10(\cos 180° + i \sin 180°)$

   (c) $3\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ (d) $5\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$

   (e) $\text{cis } 4\pi$ (f) $3 \text{ cis}(-\pi)$ (g) $5 \text{ cis } \frac{5\pi}{2}$

   (h) $2 \text{ cis } 30°$ (i) $3 \text{ cis } 135°$ (j) $5 \text{ cis } 420°$

   (k) $7 \text{ cis } 720°$ (l) $4 \text{ cis } (-60)°$ (m) $\text{cis } 225°$

   (n) $3 \text{ cis } 600°$ (o) $2 \text{ cis } \frac{7\pi}{6}$ (p) $3 \text{ cis } \frac{13\pi}{6}$

   (q) $\text{cis } 99\pi$ (r) $3 \text{ cis } 150°$ (s) $6 \text{ cis } 210°$

2. Plot each of the following complex numbers and write them in polar form using θ in the range $0° \leq \theta < 360°$.

   (a) 4 (b) $3i$ (c) -2

   (d) $-i$ (e) $1 + i\sqrt{3}$ (f) $1 - i\sqrt{3}$

   (g) $-1 + i\sqrt{3}$ (h) $-1 - i\sqrt{3}$ (i) $6 + 6i$

   (j) $-2\sqrt{3} + 2i$ (k) $-5i$ (l) $-6 - 6i\sqrt{3}$

   (m) $-3 + 3i\sqrt{3}$ (n) $-4 - 4i$ (o) 10

   (p) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (q) $-\sqrt{2}$ (r) $\sqrt{3} - i$

3. Write the general polar form of the following complex numbers.

   (a) 4 (b) -2 (c) i

   (d) $-6i$ (e) $-\sqrt{3} + i$ (f) $3\sqrt{2} - 3i\sqrt{2}$

4. Find the rectangular form of the following complex numbers.

   (a) $2 \text{ cis } 40°$ (b) $3 \text{ cis } 100°$ (c) $\text{cis } 200°$

   (d) $3 \text{ cis } 190°$ (e) $2 \text{ cis } 275°$ (f) $4 \text{ cis } 2$

5. What is the argument of the following complex numbers with $a \in Re^+$?

   (a) $a + 0i$ (b) $-a + 0i$ (c) $0 + ai$ (d) $0 - ai$

6. Explain why it is essential to use a diagram (at least a mental one) in expressing $a + bi$ in polar form.

7. Prove that if z has polar form $r \text{ cis } \theta$, then $\bar{z}$ has polar form $r \text{ cis } (-\theta)$.

8. Use the result in question (7) to give the polar form of the conjugates of the following complex numbers. Change each number to its polar form first.

   (a) 6 (b) $6i$ (c) -6 (d) $-6i$

   (e) $1 + i$ (f) $3 - 3i$ (g) $\sqrt{3} - i$ (h) $1 - i\sqrt{3}$

7.8. Multiplication and Division of Complex Numbers in Polar Form

Theorem 1. If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$, then

$$z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2) .$$

In words, "The absolute value of the product of two complex numbers is the *product* of their absolute values; the argument of the product of two complex numbers is the *sum* of their arguments."

The result of this theorem shows that multiplication of complex numbers is simpler when the complex numbers are in polar form than when they are in rectangular form.

Proof of Theorem:

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2) .$$

Therefore

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2[\cos \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2] \\ &= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 \text{ cis } (\theta_1 + \theta_2), \quad \text{as required.} \end{aligned}$$

Example 1. Find the product of 3 cis 15° and 2 cis 30° in the form $a + bi$.

Solution:

$$\begin{aligned} (3 \text{ cis } 15°)(2 \text{ cis } 30°) &= 6 \text{ cis } 45° \\ &= 6\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \\ &= 3\sqrt{2} + 3i\sqrt{2} . \end{aligned}$$

Division of complex numbers in polar form is dealt with in the following theorem.

Theorem 2. If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$, with $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2) .$$

In words, "The absolute value of a quotient of two complex numbers is the *quotient* of their absolute values; the argument of the quotient of two complex numbers is the *difference* of their arguments."

Proof of Theorem: By definition, $\dfrac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2}$ is the complex number $r \text{ cis } \theta$ such that

$$(r \text{ cis } \theta)(r_2 \text{ cis } \theta_2) = r_1 \text{ cis } \theta_1 ;$$

that is,

$$rr_2 \operatorname{cis}(\theta + \theta_2) = r_1 \operatorname{cis} \theta_1 .$$

Hence

$$rr_2 = r_1 \quad \text{and} \quad \theta + \theta_2 = \theta_1 + 2k\pi ,$$

$$r = \frac{r_1}{r_2} \quad \text{and} \quad \theta = \theta_1 - \theta_2 + 2k\pi .$$

Therefore

$$\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2), \qquad \text{as required.}$$

Example 2. Find the quotient 10 cis 105° ÷ 5 cis 75° in the form $a + bi$.

Solution:

$$\begin{aligned}\frac{10 \operatorname{cis} 105^\circ}{5 \operatorname{cis} 75^\circ} &= 2 \operatorname{cis} 30^\circ \\ &= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\ &= \sqrt{3} + i .\end{aligned}$$

EXERCISE 7.8

1. Perform the following indicated operations and write the answers in rectangular form.

   (a) $2 \operatorname{cis} 45^\circ \cdot 3 \operatorname{cis} 0^\circ$ (b) $4 \operatorname{cis} 25^\circ \cdot 2 \operatorname{cis} 65^\circ$

   (c) $2 \operatorname{cis} \pi \cdot 2 \operatorname{cis} \frac{\pi}{2}$ (d) $\operatorname{cis} 60^\circ \cdot \operatorname{cis} 120^\circ$

   (e) $4 \operatorname{cis} 0^\circ \div 2 \operatorname{cis} 45^\circ$ (f) $4 \operatorname{cis} 70^\circ \div 3 \operatorname{cis} 25^\circ$

   (g) $\operatorname{cis} \pi \div 2 \operatorname{cis} \frac{\pi}{2}$ (h) $\operatorname{cis} 60^\circ \div \operatorname{cis} 120^\circ$

   (i) $3 \operatorname{cis} 100^\circ \cdot 4 \operatorname{cis} 20^\circ$ (j) $2 \operatorname{cis} 380^\circ \div 5 \operatorname{cis} 200^\circ$

   (k) $4 \operatorname{cis} 100^\circ \cdot 3 \operatorname{cis} 35^\circ$ (l) $5 \operatorname{cis} 270^\circ \div \frac{1}{2} \operatorname{cis} 45^\circ$

2. If $z = r \operatorname{cis} \theta$, show that $z^2 = r^2 \operatorname{cis} 2\theta$.

3. Find the squares of the following complex numbers.

   (a) $3 \operatorname{cis} 30^\circ$ (b) $\sqrt{2} \operatorname{cis} 45^\circ$ (c) $3 \operatorname{cis} 60^\circ$

4. If $z = r \operatorname{cis} \theta$, $z \neq 0$, find the multiplicative inverse of z in polar form.

5. Find the multiplicative inverses of the following complex numbers.

   (a) $2 \operatorname{cis} 60^\circ$ (b) $\frac{1}{2} \operatorname{cis} (-30^\circ)$ (c) $\sqrt{2} \operatorname{cis} 135^\circ$

6. (a) If $z = \sqrt{2} \operatorname{cis} 15^\circ$, find z^4. (b) If $z = \sqrt[4]{2} \operatorname{cis} 22\frac{1}{2}^\circ$, find z^8.

7.9. De Moivre's Theorem

In question (2), Exercise 7.8, the first theorem of Section 7.8, namely, "If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$, then $z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$.", was used to prove that

$$\text{if} \quad z = r \text{ cis } \theta, \quad \text{then} \quad z^2 = r^2 \text{ cis } 2\theta\,.$$

This is a special case of a very important theorem known as De Moivre's Theorem.

De Moivre's Theorem

$$(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta, \qquad n \in C\,.$$

(Abraham de Moivre (1667-1754) was born in France but spent most of his life in England. He was a friend of Newton and made important contributions to mathematics, especially in the theory of probability.)

Proof of De Moivre's Theorem if $n \in N$: We use mathematical induction. Let S be the set of positive integers for which De Moivre's Theorem is true.

(1) $1 \in S$, since $(r \text{ cis } \theta)^1 = r \text{ cis } 1\theta$.

(2) Assume $k \in S$; that is, assume

$$(r \text{ cis } \theta)^k = r^k \text{ cis } k\theta.$$

Then

$$\begin{aligned} (r \text{ cis } \theta)^{k+1} &= (r \text{ cis } \theta)^k \cdot r \text{ cis } \theta \\ &= r^k \text{ cis } k\theta \cdot r \text{ cis } \theta \\ &= r^{k+1} \text{ cis } (k+1)\theta \qquad \text{(mult. in polar form).} \end{aligned}$$

Therefore $k + 1 \in S$ whenever $k \in S$.
Therefore $S = N$ and De Moivre's Theorem

$$(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$$

is true for all positive integers n.

De Moivre's Theorem allows us to compute powers of complex numbers written in polar form.

Example 1. Calculate $(1 - i)^{100}$.

Solution:

$$1 - i = \sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right).$$

$$\begin{aligned} (1 - i)^{100} &= (\sqrt{2})^{100} \text{ cis } (-25\pi) \\ &= 2^{50} \text{ cis } \pi \\ &= 2^{50}\,(-1 + 0i) \\ &= -2^{50}\,. \end{aligned}$$

The validity of De Moivre's Theorem for negative integers and rational numbers is demonstrated in the following examples.

Example 2. Show that $(r \text{ cis } \theta)^{-8} = r^{-8} \text{ cis } (-8\theta)$.

Solution:

$$\begin{aligned} (r \text{ cis } \theta)^{-8} &= \frac{1}{(r \text{ cis } \theta)^8} \\ &= \frac{1}{r^8 \text{ cis } 8\theta} && \text{(De Moivre's)} \\ &= \frac{1}{r^8} \text{ cis } (0 - 8\theta) && \text{(division of complex numbers)} \\ &= r^{-8} \text{ cis } (-8\theta)\,, && \text{as required.} \end{aligned}$$

Example 3. Show that $\cos \frac{2}{3}\theta + i \sin \frac{2}{3}\theta$ is a value of $(\cos \theta + i \sin \theta)^{\frac{2}{3}}$.

Solution:

$$\left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3}\right)^3 = \cos \theta + i \sin \theta$$

so that

$$\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \text{ is a value of } (\cos \theta + i \sin \theta)^{\frac{1}{3}}.$$

Also,

$$\left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3}\right)^2 = \cos \frac{2\theta}{3} + i \sin \frac{2\theta}{3}$$

so that

$$\cos \frac{2\theta}{3} + i \sin \frac{2\theta}{3} \text{ is a value of } (\cos \theta + i \sin \theta)^{\frac{2}{3}}.$$

The following example shows how De Moivre's Theorem and the Binomial Theorem can be combined to produce certain multiple angle formulae used in trigonometry.

Example 4. Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$.

Solution:

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 && \text{(De Moivre's)} \\ &= \cos^3\theta + 3i \cos^2\theta \sin \theta + 3i^2 \cos \theta \sin^2\theta + i^3 \sin^3\theta \\ & && \text{(binomial)} \\ &= (\cos^3\theta - 3 \cos \theta \sin^2\theta) + i\,(3 \cos^2\theta \sin \theta - \sin^3\theta)\,. \end{aligned}$$

Therefore

$$\cos 3\theta = \cos^3\theta - 3 \cos \theta \sin^2\theta$$

and

$$\sin 3\theta = 3 \cos^2\theta \sin \theta - \sin^3\theta, \qquad \text{as required.}$$

EXERCISE 7.9

1. Use De Moivre's Theorem to evaluate the following products; express each result in rectangular form.

   (a) $(\cos 15° + i \sin 15°)^4$
   (b) $(\cos 45° + i \sin 45°)^8$
   (c) $(\cos 20° + i \sin 20°)^{-9}$
   (d) $(\cos 15° + i \sin 15°)^{-4}$
   (e) $\left(2 \operatorname{cis} \frac{\pi}{4}\right)^2$
   (f) $\left(3 \operatorname{cis} \frac{\pi}{6}\right)^4$
   (g) $\left(\frac{1}{2} \operatorname{cis} \frac{2\pi}{3}\right)^3$
   (h) $(2 \operatorname{cis} 15°)^{-2}$
   (i) $(3 \operatorname{cis} 72°)^5$
   (j) $(3 \operatorname{cis} 90°)^{-2}$
   (k) $\frac{(1 + i)^7(-1 + i)^5}{(-1 - i)^3}$
   (l) $\frac{(\sqrt{3} + i)^4(-1 + i)^6}{(1 + i)^4}$
   (m) $(\sqrt{3} - i)^9$
   (n) i^{83}
   (o) $(-1 - i)^{-8}$
   (p) $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{-6}$
   (q) $(\sqrt{3} + i)^4(-1 + i)^6$
   (r) $\frac{(\sqrt{3} - i)^{12}(1 + i)^8}{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^6}$

2. If $A = 2 \operatorname{cis} 120°$ and $B = \frac{1}{2} \operatorname{cis} 30°$, find the following in rectangular form.

   (a) A^6
   (b) B^{10}
   (c) A^4B^5
   (d) A^9/B^6

3. Use De Moivre's Theorem and the Binomial Theorem to find formulae for the following.

   (a) $\sin 2\theta$ and $\cos 2\theta$
   (b) $\sin 4\theta$ and $\cos 4\theta$
   (c) $\sin 5\theta$ and $\cos 5\theta$
   (d) $\sin 6\theta$ and $\cos 6\theta$

4. Prove that De Moivre's Theorem is true for all negative integers n.

5. Prove that if m is rational, then $\cos m\theta + i \sin m\theta$ is one of the values of $(\cos \theta + i \sin \theta)^m$.

7.10. Roots of Complex Numbers

If u and z are complex numbers such that $u^p = z$, for some integer p, then u is a pth root of z, and we write $u = z^{1/p}$.

We may use De Moivre's Theorem to find roots of a complex number as in the following examples. We will assume that De Moivre's Theorem holds for all rational exponents.

Example 1. Find the cube root of $64i$.

Solution: To apply De Moivre's Theorem, we use the polar form of the complex number.

$$\begin{aligned} 64i &= 64\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= 64 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right), \qquad k \in I. \end{aligned}$$

If Z is a cube root of $64 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$, then

$$Z^3 = 64 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right),$$

and

$$\begin{aligned} Z &= \left[64 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)\right]^{1/3} \\ &= 4 \operatorname{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right). \end{aligned}$$

There would seem to be an infinite number of cube roots, since k can have any integral value but in fact they are not all distinct.

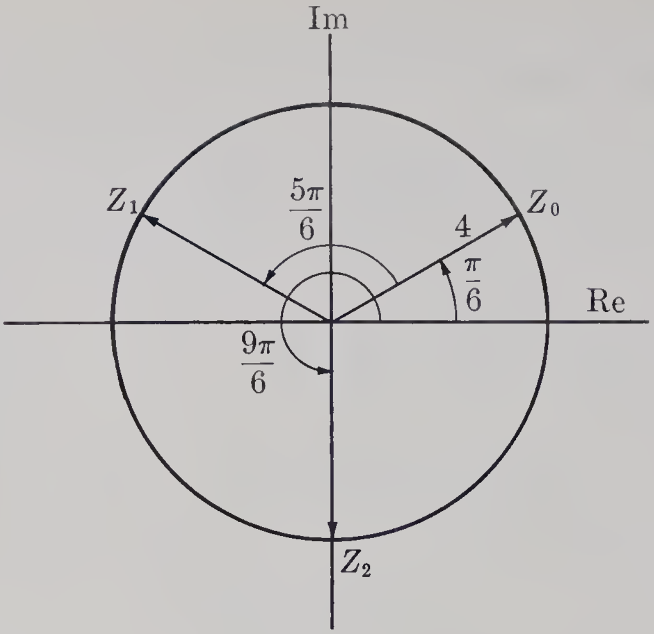
Let Z_k be the cube root corresponding to the integer k. Then

$$\begin{aligned} Z_0 &= 4 \operatorname{cis}\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2\sqrt{3} + 2i\,; \\ Z_1 &= 4 \operatorname{cis}\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -2\sqrt{3} + 2i\,; \\ Z_2 &= 4 \operatorname{cis}\frac{9\pi}{6} = 4(0 - i) = -4i\,; \\ Z_3 &= 4 \operatorname{cis}\frac{13\pi}{6} = 4 \operatorname{cis}\frac{\pi}{6} = Z_0\,; \\ Z_4 &= 4 \operatorname{cis}\frac{17\pi}{6} = 4 \operatorname{cis}\frac{5\pi}{6} = Z_1\,; \\ Z_5 &= 4 \operatorname{cis}\frac{21\pi}{6} = 4 \operatorname{cis}\frac{9\pi}{6} = Z_2\,; \qquad \text{etc.} \end{aligned}$$

Thus $64i$ has three distinct cube roots since, for any integer t,

$$Z_{3t} = Z_0, \qquad Z_{3t+1} = Z_1, \qquad \text{and} \qquad Z_{3t+2} = Z_2\,.$$

The three distinct cube roots of $64i$ may be plotted on the Argand plane; the points representing these cube roots lie on a circle about the origin of radius 4 and the vectors representing these cube roots divide this circle into three equal sectors.



The three complex numbers Z_0, Z_1, and Z_2 of Example 1 are distinct solutions of the polynomial equation

$$Z^3 - 64i = 0\,.$$

We may call such an equation, a pure equation of degree 3. By a pure equation of degree n, we mean an equation of the form

$$z^n - A = 0\,,$$

where A is a complex number.

Example 2. Find the solution set of the equation $z^4 - 16 = 0$.

Solution: We require the fourth roots Z of 16. Write

$$16 = 16 \text{ cis } (0 + 2k\pi)\,;$$

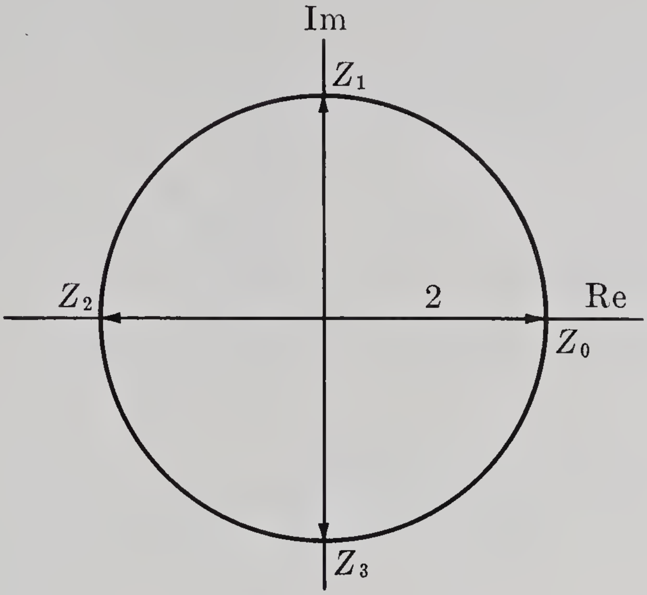
then

$$\begin{aligned} Z &= [16 \text{ cis } (0 + 2k\pi)]^{1/4} \\ &= 2 \text{ cis } \frac{k\pi}{2}\,, \end{aligned}$$

and therefore

$$\begin{aligned} Z_0 &= 2 \text{ cis } 0 = 2\,, \\ Z_1 &= 2 \text{ cis } \frac{\pi}{2} = 2i\,, \\ Z_2 &= 2 \text{ cis } \pi = -2\,, \\ Z_3 &= 2 \text{ cis } \frac{3\pi}{2} = -2i\,. \end{aligned}$$

Again, $Z_4 = Z_0$, $Z_5 = Z_1$, $Z_6 = Z_2$, $Z_7 = Z_3$, etc., so that there are four distinct fourth roots of 16. If we plot these fourth roots on the Argand plane, the corresponding points lie on a circle, centre the origin, radius 2, and the corresponding vectors divide the circle into four equal sectors.



The solution set of $z^4 - 16 = 0$ is $\{2, 2i, -2, -2i\}$.

In general, to find the pth roots Z of a complex number z, we write z in general polar form

$$z = r \text{ cis } (\theta + 2k\pi),$$

and apply De Moivre's Theorem to obtain

$$\begin{aligned} Z &= [r \text{ cis } (\theta + 2k\pi)]^{1/p} \\ &= r^{1/p} \text{ cis } \left(\frac{\theta + 2k\pi}{p}\right). \end{aligned}$$

Note that $r^{1/p}$ is the real positive pth root of the positive real number r. The p distinct roots $Z_0, Z_1, \cdots, Z_{p-1}$ are obtained by replacing k in turn by $0, 1, 2, \cdots, p - 1$.

EXERCISE 7.10

1. Find the following indicated roots; leave the answers in polar form if they involve angles θ for which $\cos \theta$ and $\sin \theta$ are not well known. Represent the roots graphically.
   (a) square roots of $9i$
   (b) square roots of $-4i$
   (c) square roots of 16 cis 120°
   (d) cube roots of 8 cis 180°
   (e) cube roots of 27
   (f) cube roots of $-8i$

(g) sixth roots of 81 cis 180°
(h) fifth roots of 32 cis 150°
(i) sixth roots of 1
(j) fourth roots of -64
(k) cube roots of -1
(l) fourth roots of $-8 + 8i\sqrt{3}$
(m) sixth roots of i
(n) fifth roots of $-4(1 - i)$
(o) square roots of $1 - i\sqrt{3}$
(p) eighth roots of $\frac{1}{2}(i\sqrt{3} - 1)$

2. Find the solution sets of the following equations.
(a) $z^3 = 64$
(b) $z^2 = 16i$
(c) $z^4 + 81 = 0$
(d) $z^5 = 32$
(e) $z^5 + 1 = 0$
(f) $z^2 + 1 - i\sqrt{3} = 0$
(g) $z^4 + 4 + 4i\sqrt{3} = 0$
(h) $z^3 = 2 + i$

3. Use tables to find approximations to the square roots of
(a) $3 + 4i$
(b) $5 - 12i$

4. Find approximate roots of the equation

$$z^2 - (5 + i)z + 4 - \frac{i}{2} = 0\,.$$

Chapter Summary

Complex number · Numeral for a complex number in the form $a + bi$ with $i^2 = -1$, a, b real, or (a, b) with a, b real · Imaginary and real coefficient or component

Geometric representation of complex numbers · Argand plane

Absolute value or modulus or length of a complex number

Equality of complex numbers

Addition of complex numbers

Properties of addition: commutative, associative, existence of identity, existence of negatives

Multiplication of complex numbers

Properties of multiplication: commutative, associative, existence of identity, existence of inverses for nonzero complex numbers, distributivity

Complex conjugates

Division of complex numbers

Solution of quadratic equations with complex roots

Polar form of a complex number

Multiplication and division of complex numbers in polar form

De Moivre's Theorem

Roots of complex numbers

REVIEW EXERCISE 7

1. State the real and imaginary coefficients of the following complex numbers,
   (a) $3 + 5i$ (b) $4 - i$ (c) $-\frac{1}{2} + 2i$ (d) $5 + 0i$
   and represent them by points in the Argand plane.

2. Give the absolute values of the complex numbers in question (1).

3. If $z_1 = 3 - 2i$ and $z_2 = -4 + 5i$, find the following numbers and their absolute values.
   (a) $z_1 + z_2$ (b) $z_2 + z_1$ (c) $z_1 - z_2$ (d) $z_2 - z_1$

4. If $z_1 = -4 - 3i$, $z_2 = -1 + 5i$, and $z_3 = 0 - 4i$, find the following.
   (a) $(z_1 + z_2) + z_3$ (b) $z_1 + (z_2 + z_3)$ (c) $z_1 - z_2 - z_3$

5. Solve for z.
   (a) $(3 - 5i) + z = -2 + 4i$. (b) $5 - 3i = z + (4 - 2i)$.

6. Represent geometrically, the complex numbers $z_1 = 5 - 2i$, $z_2 = -3 + 4i$, and $z_1 + z_2$, and show that the three points used, together with the origin, form a parallelogram.

7. If $z_1 = 5 + 2i$, $z_2 = -3 + i$, $z_3 = -4 - 3i$, find the following.
   (a) $z_1 z_2$ (b) $z_2 z_1$ (c) $(z_1 z_2) z_3$ (d) $z_1(z_2 z_3)$
   (e) $z_1(z_2 + z_3)$ (f) $z_1 z_2 + z_1 z_3$ (g) $(z_1 + z_2) z_3$ (h) $z_1 z_3 + z_2 z_3$

8. For the complex numbers of question (7), find the following.
   (a) $z_1 + \overline{z_1}$ (b) $z_2 + \overline{z_2}$ (c) $z_3 + \overline{z_3}$ (d) $z_1 \overline{z_1}$
   (e) $z_2 \overline{z_2}$ (f) $z_3 \overline{z_3}$ (g) $\overline{z_1 + z_2}$ (h) $\overline{z_1} + \overline{z_2}$
   (i) $\overline{z_1 z_2}$ (j) $\overline{z_1}\,\overline{z_2}$ (k) $\overline{z_1 z_2 z_3}$ (l) $\overline{z_1}\,\overline{z_2}\,\overline{z_3}$

9. For the complex numbers of question (7), find the following
   (a) $\dfrac{z_1}{z_2}$ (b) $\dfrac{z_2}{z_3}$ (c) $\dfrac{z_3}{z_1}$ (d) $\dfrac{z_1}{z_2} \cdot \dfrac{z_2}{z_3} \cdot \dfrac{z_3}{z_1}$
   (e) $\dfrac{1}{z_1}$ (f) $\dfrac{1}{z_2}$ (g) $\dfrac{1}{z_3}$ (h) $\dfrac{1}{z_1\, z_2\, z_3}$

10. Solve the following equations for z.
    (a) $(4 - 3i)z = 5 + 2i$. (b) $(-3 + 2i)z = 6$.

11. Plot the following complex numbers and give their rectangular form.
    (a) 4 cis 0° (b) 3 cis 225° (c) 2 cis 210° (d) 4 cis 100π

12. Plot the following complex numbers and give their polar form $(0° \leq \theta < 360°)$.
    (a) 6 (b) $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$ (c) $-3\sqrt{3} + 3i$ (d) $\sqrt{3} + i$

13. Perform the indicated operations and give the answer in rectangular form.
    (a) $5 \text{ cis } 30° \cdot 3 \text{ cis } 60°$ (b) $6 \text{ cis } 85° \div 3 \text{ cis } 40°$
    (c) $(\sqrt{2} \text{ cis } 15°)^4$ (d) $(\sqrt{3} \text{ cis } 30°)^3$
    (e) $\dfrac{1}{2 \text{ cis } 30°}$ (f) $\dfrac{3}{4 \text{ cis } 45°}$

14. Use De Moivre's Theorem to evaluate the following.
    (a) $(1 - i)^9$ (b) $(\sqrt{3} + i)^8$

15. Find the following indicated roots and represent the roots graphically.
    (a) cube roots of $27i$ (b) fifth roots of $8 - 8i\sqrt{3}$

16. Find the solution sets of the following equations.
    (a) $z^4 + 81i = 0$ (b) $z^4 - 4 + 4i\sqrt{3} = 0$

17. Verify algebraically that
    (a) $(r \text{ cis } \theta)^{-5} = \dfrac{1}{r^5} \text{ cis } (-5\theta)$,
    (b) $\text{cis } \frac{5}{2}\theta$ is one of the values of $(\text{cis } \theta)^{\frac{5}{2}}$,
    (c) $\text{cis } (-\frac{3}{4}\theta)$ is one of the values of $(\text{cis } \theta)^{-\frac{3}{4}}$.

Chapter 8

AN INTRODUCTION TO ALGEBRAIC SYSTEMS

8.1. Introduction

In this chapter we introduce the student to a more general view of mathematical systems.

We begin our study by considering sets of elements other than numbers, primarily sets consisting of mappings or transformations. We then consider combinations of such set elements by the introduction of binary operations with examples from different topics in algebra and geometry. This leads us to a study of the properties of such binary operations. Finally, by examining examples of such sets with operations, we see certain patterns emerging which lead to generalizations producing algebraic systems known as groupoids, semigroups, groups, rings, and fields. Indeed, we have already examined the pattern of the properties of the sets *Ra*, *Re*, and *C* under the operations of addition and multiplication, and we have noted that these systems have a common core of properties which we labelled as "field properties". These number field systems are special cases of the more general situation.

EXERCISE 8.1

1. List the field properties associated with the sets *Ra*, *Re*, and *C*.

8.2. Sets of Permutations

Suppose that a set of three books A, B, and C is sitting in this order on a shelf and that a student passes by and interchanges books B and C. Originally the books stood ABC; now they stand ACB. The change in order or rearrangement effected by the student is an example of a mapping of the set {A, B, C} of books onto itself; such a mapping is called a permutation. In this mapping, book B is replaced by (changed to, transformed into, mapped onto) book C, book C is replaced by book B, and book A is unchanged (or mapped onto itself). We indicate this mapping or permutation by the symbol

$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$

in which the first line gives the original arrangement of the books and the second line gives the rearrangement after the mapping or transformation or permutation has been applied. Such a mapping is a function; the top line gives the domain and the bottom line, the range. (They are the same set for a permutation.)

We know from our earlier work that 3! or six such permutations are possible:

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}, \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}, \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}, \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}, \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}, \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}.$$

The first of these permutations is the mapping of the set of books in which each book is left in its original position; this is the identity or neutral permutation.

Note that we would obtain exactly the same set of permutations if we were rearranging a set of pictures of three N.H.L. hockey players or the names of three Canadian cities. The elements of the set being rearranged are quite immaterial but, of course, the number of elements in the set being rearranged will be important in determining the permutations and the number of such permutations. A set {A, B, C, D} of four books would give rise to a set of 4! or 24 possible permutations.

Example 1. Write permutations of the set {A, B, C, D} in which A and D are (a) interchanged, (b) unchanged.

Solution: (a) The required permutations have the form

$$\begin{pmatrix} A & B & C & D \\ D & * & * & A \end{pmatrix}$$

There are only two such permutations,

$$\begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}.$$

(b) The required permutations are

$$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}.$$

Example 2. Write the permutations of the set {A, B, C, D, E} in which A, C, and D are unchanged.

Solution: The required permutations are of the form

$$\begin{pmatrix} A & B & C & D & E \\ A & * & C & D & * \end{pmatrix}.$$

There are only two such permutations,

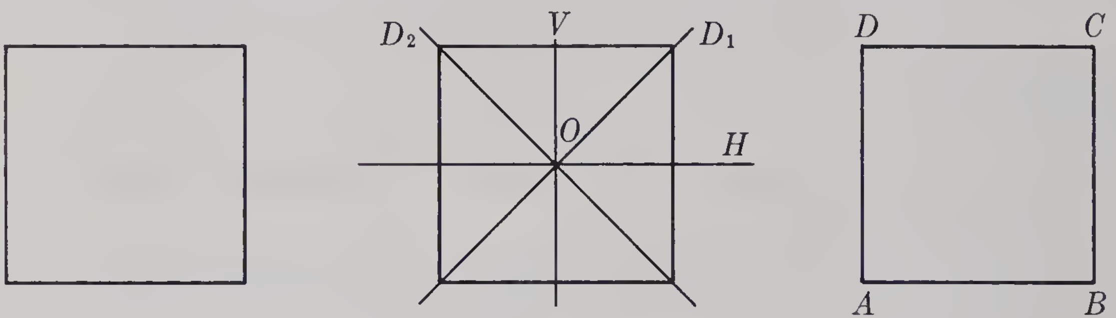
$$\begin{pmatrix} A & B & C & D & E \\ A & B & C & D & E \end{pmatrix} \text{ and } \begin{pmatrix} A & B & C & D & E \\ A & E & C & D & B \end{pmatrix}.$$

EXERCISE 8.2

1. How many permutations are there on a set of two books? Write symbols for these permutations.
2. Write the set of permutations on three objects, using the symbols 1, 2, and 3 for the objects.
3. Write the set of permutations of the set {A, B, C, D}.
4. Write the permutations of the set {1, 2, 3, 4} in which 2 and 4 are interchanged.
5. Give the permutations of the set {A, B, C, D, E} in which
   (a) A and E are unchanged,
   (b) A and E are interchanged,
   (c) B, C, and D are unchanged,
   (d) B is changed into C, C into D, and D into B.
6. Give the permutations of the set {1, 2, 3, 4, 5} in which
   (a) 1, 2, and 4 are unchanged,
   (b) 3 and 5 are unchanged,
   (c) 4 and 5 are interchanged,
   (d) 1 and 3 are interchanged and 2 and 5 are interchanged,
   (e) 1 is changed into 3, 3 into 5, and 5 into 1.
7. Write the permutations of the set {1, 2, 3, 4, 5, 6} in which
   (a) 4, 5, and 6 are unchanged,
   (b) 1, 2, and 3 are unchanged,
   (c) 1 and 4, 2 and 5, 3 and 6 are interchanged,
   (d) 1 and 6, 2 and 5 are interchanged.
8. How many permutations on n distinct symbols are there which leave two given symbols unchanged?

8.3. Sets of Motions

Consider the square shown in Figure 8.1 (*a*). The lines H, V, D_1, and D_2 are drawn in (*b*) for convenience in our discussion. It may prove useful to construct a square out of cardboard. If the square were rotated through 90° counterclockwise about O while we were not looking, we would be unable to tell anything had been changed. (Try it!) In carrying out that rotation, we have mapped the square onto itself. Alternatively, we can consider the situation in which the square of cardboard is fitted into a corresponding square hole. If the square is picked up and rotated through 90°, it can be fitted back into the hole.



(*a*) (*b*) (*c*)

Figure 8.1

We can consider the set of rigid motions which move the square in such a way that it can be put back into the square hole. In these cases we can tell that the square has been moved only if it is marked in some way. We list these motions.

(1) R_{90}, a counterclockwise rotation of 90° about O

(2) R_{180}, a counterclockwise rotation of 180° about O

(3) R_{270}, a counterclockwise rotation of 270° about O

(4) H, a rotation of 180° about the line H

(5) V, a rotation of 180° about the line V

(6) D_1, a rotation of 180° about the line D_1

(7) D_2, a rotation of 180° about the line D_2

Finally, we include the identity or neutral motion.

(8) R_0, a counterclockwise rotation of 0° about O

(Note that a rotation of 360° replaces the square in its original position; thus, R_{360} is identical with R_0.) This set

$$\{R_0, R_{90}, R_{180}, R_{270}, H, V, D_1, D_2\}$$

is called the set of motions of the square.

Note that a rotation of 180° about line H involves rotating the square in 3-space. Try it with a cardboard model; observe that such a motion is impossible if the cardboard must remain on a table top (that is, in a plane). If we wished to consider the square as remaining in a plane, we would obtain the same result as motion H by considering a reflection of the square in the line H.

All of the above motions can be described as permutations of the vertices A, B, C, and D of the square (Figure 8.1 (c)). For example, R_{270} will correspond to permutation

$$\begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}.$$

We use the equals sign here to mean "corresponds to" and write

$$R_{270} = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}.$$

Similarly,

$$V = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}.$$

Of course we will obtain only eight such permutations in this way, so that not every permutation on $\{A, B, C, D\}$ will correspond to a rigid motion of the square.

We will see in the exercises that the number of elements in the set of rigid motions of a geometrical figure depends on the symmetry possessed by the figure.

EXERCISE 8.3

What rigid motions carry the following figures into themselves? State the corresponding permutations in each case. (A diagram is essential.)

1. A triangle of three unequal sides
2. An isosceles triangle
3. A quadrilateral with four unequal sides
4. An equilateral triangle
5. A nonsquare rectangle
6. A nonsquare rhombus
7. A square
8. A regular pentagon
9. A regular hexagon
10. A regular octagon
11. How many permutations are there corresponding to the motions of a regular polygon of n sides?
12. Which permutations on $\{A, B, C, D\}$ do not correspond to motions of the square?

8.4. Some Additional Sets

In addition to the sets of permutations and motions that we have just studied, and the sets of numbers that we have studied earlier, there are sets of other types of elements that will prove useful to us as examples.

(1) *The Set of Translations in the Plane*

The transformation performed by the mapping

$$(x, y) \to (u, v)\,,$$

with $u = x + h$ and $v = y + k$, is a translation of the plane. We may use the symbol $T_{h,k}$ to refer to such a transformation and write

$$T_{h,k}\colon (x, y) \to (x + h,\ y + k)\,,$$

or, since this merely indicates that the image of x is $x + h$ and that of y is $y + k$, we often write

$$T_{h,k}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \end{bmatrix}$$

to indicate the same thing. The latter is a useful notation when we study matrices. Thus, for example, $T_{2,5}$ is the transformation under which

$$(x, y) \to (x + 2,\ y + 5)\,,$$

that is,

$$T_{2,5}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2 \\ y + 5 \end{bmatrix}.$$

The set consisting of all such translations $T_{h,k}$ will contain an infinite number of elements. It will contain the identity or neutral translation $T_{0,0}$ under which

$$(x, y) \to (x, y)\,.$$

(2) *The Set of Rotations in the Plane*

The transformation performed by the mapping

$$(x, y) \to (u, v)\,,$$

with $u = x\cos\theta - y\sin\theta$ and $v = x\sin\theta + y\cos\theta$, is a rotation of the plane through angle θ measured counterclockwise about the origin. We may use the symbol $R_{(\theta)}$ to refer to such a rotation. Thus, $R_{(\pi/4)}$ is the transformation under which

$$(x, y) \to \left(\frac{x - y}{\sqrt{2}},\ \frac{x + y}{\sqrt{2}}\right).$$

We write

$$R_{(\pi/4)} : (x, y) \to \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$$

or

$$R_{(\pi/4)}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{x-y}{\sqrt{2}} \\ \dfrac{x+y}{\sqrt{2}} \end{bmatrix}.$$

Again, the set of all such rotations $R_{(\theta)}$ will contain an infinite number of elements. It will contain the identity or neutral rotation $R_{(0)}$ under which

$$(x, y) \to (x, y).$$

(3) *A Set of Reflections*

The finite set $\{I, R_x, R_y, R_{xy}\}$ consisting of the following four reflections in the plane will be used as an example in this chapter.

I: $(x, y) \to (x, y)$; the identity or neutral mapping

R_x: $(x, y) \to (x, -y)$; reflection in the x-axis

R_y: $(x, y) \to (-x, y)$; reflection in the y-axis

R_{xy}: $(x, y) \to (-x, -y)$; reflection in both the x- and y-axes, that is, reflection in the origin

We will also write

$$R_x\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

etc. There should be no confusion between these symbols for reflections and the symbol $R_{(\theta)}$ for rotations. The reader is encouraged to draw some diagrams illustrating the effect of these reflections on various geometrical figures.

(4) *Sets of Linear Transformations*

The elements in (2) and (3) above are special cases of the more general type of mapping in the plane defined by

$$(x, y) \to (ax + by, cx + dy).$$

These mappings are known as linear transformations. Thus, for example, the rotation $R_{(\pi/4)}$ given in (2) above is a linear transformation with

$$a = -b = c = d = \frac{1}{\sqrt{2}}.$$

(5) *Sets of Functions*

Sets of functions defined on a certain domain will be used as examples in this chapter. We have already pointed out that permutations are functions, as, indeed, are the transformations we have described.

(6) *Sets of* 2×2 *Real Matrices*

A 2×2 real matrix is a rectangular array consisting of two rows and two columns of real numbers.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We use a_{ij} to denote the entry or component in row i and column j. Sets of 2×2 real matrices will be used as examples in this chapter.

EXERCISE 8.4

1. Write the mappings corresponding to the following symbols.
   (a) $T_{3,4}$ (b) $T_{-2,1}$ (c) $T_{0,0}$ (d) $T_{0,-3}$ (e) $T_{-3,-3}$
   (f) $R_{(\pi/3)}$ (g) $R_{(\pi/2)}$ (h) $R_{(\pi)}$ (i) $R_{(-\pi/4)}$ (j) $R_{(3\pi/2)}$
2. State the transformations which translate the points with co-ordinates
   (a) $(3, 2)$ (b) $(-5, 2)$ (c) $(-4, 0)$ (d) $(2, -3)$ (e) $(-3, -3)$
   to the origin.
3. State the transformations which translate the points with integral co-ordinates on the following ellipse to the origin.
$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$
4. State the transformations which rotate the plane so that the points with the following co-ordinates lie on the x-axis.
   (a) $(1, \sqrt{3})$ (b) $(-1, \sqrt{3})$ (c) $(-1, -\sqrt{3})$ (d) $(1, -\sqrt{3})$
5. State the transformations which rotate the plane so that the points with the following co-ordinates lie on the y-axis.
   (a) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (b) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (c) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ (d) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
6. State the transformations representing the linear functions
$$f_{a,b}: x \rightarrow ax + b,$$
   if $a, b \in I$ and $a^2 + b^2 = 4$.
7. Write the set of all 2×2 real matrices with $a_{12} = a_{21} = 0$ and $a_{11}, a_{22} \in I$, where $a_{11}{}^2 + a_{22}{}^2 \leq 9$.

8.5. Binary Operations

A binary operation Θ on the elements of a set $\{a, b, c, \cdots\}$ is an operation that associates with each pair of elements of the set a unique element to which we give the symbol $a \Theta b$. (Symbols such as $a \circ b$, $a \times b$, etc., may also be used.) In considering pairs of elements of the set, we will consider a and a, b and b, c and c, etc., as suitable pairs so that we will have elements $a \Theta a$, $b \Theta b$, $c \Theta c$, etc., formed by the binary operation.

It sometimes proves useful to think of a binary operation as a meat-grinding machine with two openings at the top into which elements can be stuffed and a single opening at the bottom from which the result emerges. This emphasizes that a binary operation acts on *pairs* of elements to produce in each case a single element. Note also that the machine has a left opening and a right opening so that both $a \Theta b$ and $b \Theta a$ are produced, and there is no reason to assume that these are the same element.

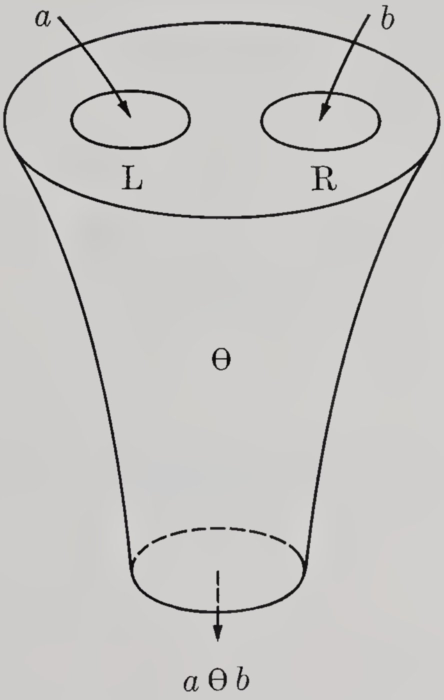


Figure 8.2

Binary Operation Machine

We are already familiar with the binary operations of addition and multiplication of real numbers. Indeed we are so familiar with these operations that their application is automatic. This sometimes clouds the fact that these operations are binary. Given 5, 6, and 7 to add, for example, we may lose sight of the fact that the addition is performed mentally on two integers at a time: $(5 + 6) + 7$, $5 + (6 + 7)$, etc.

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In this section, and the next two, we wish to consider binary operations on the elements of the sets we have just studied, sets of permutations, motions, etc. In so doing, we normally will refer to the operation Θ as the "product operation" and to $a \Theta b$ as the "product" of a and b, even though Θ may bear absolutely no relation to the multiplication of real numbers.

The following example indicates the possibility of binary operations on real numbers apart from ordinary addition and multiplication.

Example 1. Find the following products if $a \Theta b$ is defined to be the greatest common factor of a and b.

(a) $6 \Theta 9$ (b) $4 \Theta 7$ (c) $-5 \Theta 5$

Solution:

(a) $$6 \Theta 9 = 3.$$

(b) $$4 \Theta 7 = 1.$$

(c) $$-5 \Theta 5 = 5.$$

Example 2. Find the following products if $a \Theta b = a + ab$.

(a) $5 \Theta 3$ (b) $3 \Theta 5$ (c) $3 \Theta 3$

Solution:

(a) $$5 \Theta 3 = 5 + 5(3) = 20.$$

(b) $$3 \Theta 5 = 3 + 3(5) = 18.$$

(c) $$3 \Theta 3 = 3 + 3(3) = 12.$$

Note that $5 \Theta 3 \neq 3 \Theta 5$ for this binary operation Θ.

Binary operations called addition and multiplication can be defined for 2×2 real matrices. Addition is defined in terms of the addition of corresponding components.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

The definition for multiplication is more complicated.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \end{bmatrix}$$
$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

As an aid to remembering this definition, note that c_{ij}, the component in row i and column j of the product matrix, is the sum of the products of corresponding elements of row i of the left matrix and column j of the right matrix.

$$c_{11} = a_{11}\ b_{11} + a_{12}\ b_{21} = \underset{\textit{row } 1}{(a_{11} \quad a_{12})}\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \textit{ column } 1$$

$$c_{12} = a_{11}\ b_{12} + a_{12}\ b_{22} = \underset{\textit{row } 1}{(a_{11} \quad a_{12})}\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} \textit{ column } 2$$

etc.

Example 3. Find the sum and product of the matrices

$$A = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}.$$

Solution:

$$A + B = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 - 3 & -1 + 1 \\ -2 + 5 & 3 + 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} (4)(-3) + (-1)(5) & (4)(1) + (-1)(2) \\ (-2)(-3) + (3)(5) & (-2)(1) + (3)(2) \end{bmatrix}$$
$$= \begin{bmatrix} -17 & 2 \\ 21 & 4 \end{bmatrix}.$$

EXERCISE 8.5

1. Find (i) $5 \Theta 2$ (ii) $2 \Theta 5$ (iii) $3 \Theta 6$
(iv) $5 \Theta 5$ (v) $(2 \Theta 5) \Theta 3$ (vi) $6 \Theta (2 \Theta 3)$

for the binary operations defined as follows.

(a) $a \Theta b = a - b$.
(b) $a \Theta b = a^2 + b^2$.
(c) $a \Theta b = (a + b)^2$.
(d) $a \Theta b = 2a + 3b$.
(e) $a \Theta b = \frac{a}{b}$.
(f) $a \Theta b =$ maximum of a and b.
(g) $a \Theta b =$ l.c.m. of a and b.
(h) $a \Theta b = a$.
(i) $a \Theta b = b$.
(j) $a \Theta b = \frac{a + b}{2}$.
(k) $a \Theta b = a^b$.
(l) $a \Theta b = a - ab$.
(m) $a \Theta b =$ remainder when a is divided by b.
(n) $a \Theta b =$ remainder when b is divided by a.

2. Find $A + B$, $B + A$, AB, and BA in the following.

(a) $A = \begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}$, $\quad B = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -5 \\ 4 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 4 & 1 \\ 5 & 3 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & -1 \\ 5 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(g) $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$, $\quad B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

8.6. Products of Permutations • Products of Motions

Consider the two permutations

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Both of these permutations produce rearrangements of the four symbols 1, 2, 3, 4. Now consider the result of first performing permutation P and then permutation Q. We trace the result on the symbols of such a combination of permutations.

$$\begin{array}{ccccc} & \text{under } P & & \text{under } Q & \\ 1 & \longrightarrow & 4 & \longrightarrow & 3 \\ 2 & \longrightarrow & 1 & \longrightarrow & 2 \\ 3 & \longrightarrow & 2 & \longrightarrow & 1 \\ 4 & \longrightarrow & 3 & \longrightarrow & 4 \end{array}$$

The overall result of this combination of permutations is thus equal to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

We define this permutation to be the product $P \Theta Q$ of the permutations P and Q (in that order).

DEFINITION. The product $P \Theta Q$ of two permutations P and Q on the same symbols is the single permutation equal to the successive performance of permutations P and Q (P first, Q second).

Example 1. Find the product $Q \Theta P$ for the permutations above.

Solution: We write

$$Q \Theta P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Theta \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

and trace the "fate" of each symbol as a result of the successive application of Q and P. This is normally done mentally.

$$\begin{array}{ccccc} 1 & \longrightarrow & 2 & \longrightarrow & 1 \\ 2 & \longrightarrow & 1 & \longrightarrow & 4 \\ 3 & \longrightarrow & 4 & \longrightarrow & 3 \\ 4 & \longrightarrow & 3 & \longrightarrow & 2 \end{array}$$

Therefore

$$Q \Theta P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

We note that $Q \Theta P \neq P \Theta Q$ so that this binary operation, multiplication or combination of permutations, is not always commutative.

Example 2. Find $P^2 = P \Theta P$ and $P^2 \Theta P$ if

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}.$$

Solution:

$$P \Theta P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} \Theta \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}.$$

Therefore

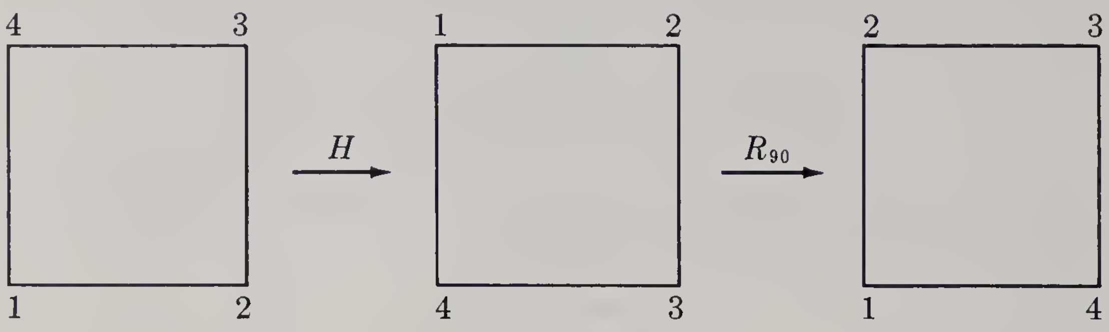
$$P^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}.$$

$$P^2 \Theta P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix} \Theta \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

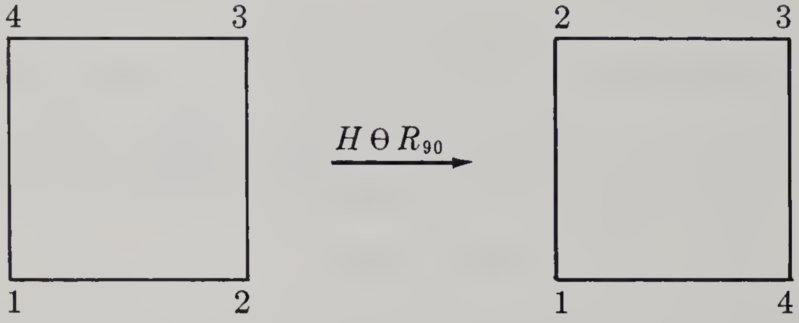
We have already seen that motions of rectilinear figures (squares, triangles, etc.) may be studied by studying the permutations of the vertices of these figures. Thus it is natural, in the light of our definition of the product of two permutations, to define the product $M_1 \Theta M_2$ of two motions M_1 and M_2 to be the single motion equal to the application of motion M_1 followed by the application of motion M_2.

Example 3. Find $H \Theta R_{90}$ and $R_{90} \Theta H$ for the motions H and R_{90} of the square. (See Section 8.3.)

Solution: By definition, $H \Theta R_{90}$ is the result of carrying out motion H, and then motion R_{90}.

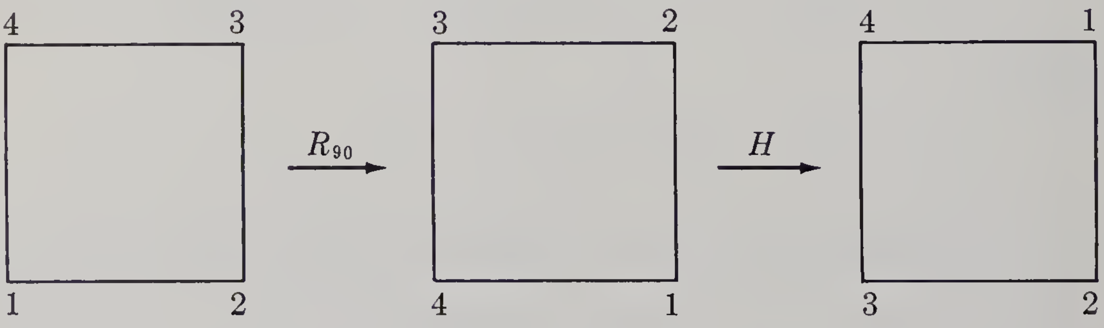


Thus $H \Theta R_{90}$ is the motion



so that $H \Theta R_{90} = D_1$.

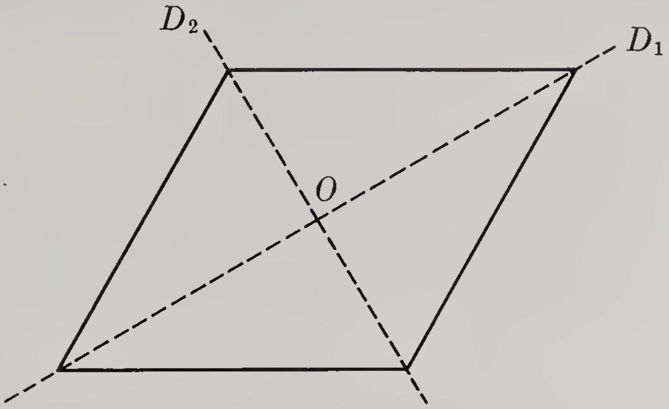
Similarly,



so that $R_{90} \Theta H = D_2$.

Again we see that $H \Theta R_{90} \neq R_{90} \Theta H$, so that multiplication of motions is not necessarily commutative.

Example 4. Set up a multiplication table for the motions R_0, R_{180}, D_1, D_2 of the rhombus which are as indicated in the diagram.



Solution: R_0 and R_{180} are counterclockwise rotations of 0° and 180° about O and D_1 and D_2 are reflections in the diagonals D_1 and D_2.

The following multiplication table may be checked.

Θ	R_0	R_{180}	D_1	D_2
R_0	R_0	R_{180}	D_1	D_2
R_{180}	R_{180}	R_0	D_2	D_1
D_1	D_1	D_2	R_0	R_{180}
D_2	D_2	D_1	R_{180}	R_0

Note that the entries in the table follow the pattern shown below in which the first motion is in the left column and the second is in the top row.

Θ	P
·	·
·	·
·	·
Q	· · · $Q \Theta P$ · · ·
·	·
·	·
·	·

EXERCISE 8.6

1. Find $P \Theta Q$, $Q \Theta P$, P^2, and Q^2 for the following permutations.

   (a) $P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

   (b) $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$.

   (c) $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$.

   (d) $P = \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \\ \mathrm{c} & \mathrm{e} & \mathrm{d} & \mathrm{a} & \mathrm{b} \end{pmatrix}$, $Q = \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \\ \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{a} \end{pmatrix}$.

   (e) $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 2 & 3 & 8 & 7 & 9 & 6 & 5 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 7 & 3 & 1 & 4 & 8 & 6 & 9 \end{pmatrix}$.

2. Find $(P \Theta Q) \Theta R$ and $P \Theta (Q \Theta R)$ in the following.

   (a) P and Q from question (1a), $R = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

   (b) P and Q from question (1b), $R = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$.

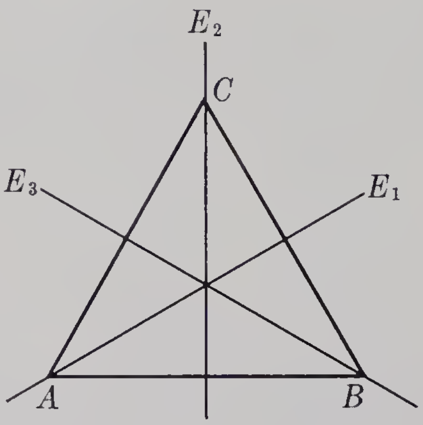
   (c) P and Q from question (1c), $R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$.

   (d) P and Q from question (1d), $R = \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \\ \mathrm{b} & \mathrm{c} & \mathrm{a} & \mathrm{e} & \mathrm{d} \end{pmatrix}$.

3. From the results of question (2), draw a conclusion regarding the multiplication of permutations.

4. If E_1, E_2, and E_3 refer to the reflections of the equilateral triangle ABC as shown, and R_0, R_{120}, R_{240}, to the usual rotations about the centre, find

   (a) $E_1 \Theta R_{120}$, (b) $E_2 \Theta E_3$, (c) $R_{240} \Theta E_2$.



5. The symbols in (a) to (l) refer to motions of the square (Section 8.3). Find the following products.

(a) $R_{90} \ominus R_{180}$	(b) $R_{180} \ominus R_{90}$	(c) $R_{90} \ominus R_{90}$
(d) $R_{180} \ominus R_{180}$	(e) $H \ominus V$	(f) $V \ominus H$
(g) $R_{90} \ominus D_1$	(h) $D_1 \ominus R_{90}$	(i) $D_1 \ominus D_2$
(j) $D_2 \ominus D_1$	(k) $D_1 \ominus V$	(l) $H \ominus R_{270}$

6. Let $\{I, R_x, R_y, R_{xy}\}$ be the set consisting of the identity mapping I and the reflections about the x-axis, the y-axis, and the origin, respectively. (See Section 8.4.) Complete the following multiplication table in which $R_x \ominus R_y$ means a reflection in the x-axis followed by a reflection in the y-axis, etc.

$\ominus$	I	R_x	R_y	R_{xy}
I	I	R_x	*	*
R_x	*	*	*	*
R_y	R_y	*	*	*
R_{xy}	*	*	*	I

8.7. Products of Transformations • Products of Functions

In this section we will introduce a binary operation on the elements of a set of transformations.

It often happens that a mapping or transformation T can be followed by a second transformation S and the overall result is equal to the performance of a single mapping which, by definition, we call the product of the two transformations.

DEFINITION. The product ST of two transformations S and T is the single transformation which is equal to performing first T and then S.

Note that there is no misprint here; ST indicates T first, S second. The reason for using this order in the symbol will be seen in our examples. Symbols such as $S \ominus T$ or $S \circ T$ could also be used to indicate this product, but ST seems to be more convenient.

We have used the phrase "transformation which is equal to" in our definition. We should point out that, by definition, two transformations are equal if they are mappings of the same set and if each element of the set has the same image under the two transformations. Thus, if $T(a)$ represents the image of element a under transformation T, then by definition, $T = S$ if T and S are both mappings of a set A and $T(a) = S(a)$ for all $a \in A$.

Example 1. Find $T_{2,3}\, T_{4,6}$ if

$$T_{2,3}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y+3 \end{bmatrix} \quad \text{and} \quad T_{4,6}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+4 \\ y+6 \end{bmatrix}, \qquad x,\, y \in Re.$$

Solution: By definition,

$$\begin{aligned}
(T_{2,3}\, T_{4,6})\begin{bmatrix} x \\ y \end{bmatrix} &= T_{2,3}\left(T_{4,6}\begin{bmatrix} x \\ y \end{bmatrix}\right) && \text{(Perform } T_{4,6} \text{ first.)} \\
&= T_{2,3}\begin{bmatrix} x+4 \\ y+6 \end{bmatrix} && \text{(Then perform } T_{2,3}\text{.)} \\
&= \begin{bmatrix} (x+4)+2 \\ (y+6)+3 \end{bmatrix} \\
&= \begin{bmatrix} x+(4+2) \\ y+(6+3) \end{bmatrix} \\
&= \begin{bmatrix} x+6 \\ y+9 \end{bmatrix} = T_{6,9}\begin{bmatrix} x \\ y \end{bmatrix}.
\end{aligned}$$

Therefore

$$T_{2,3}\, T_{4,6} = T_{6,9}.$$

The result of performing the translation $T_{4,6}$ first, followed by the translation $T_{2,3}$, is equal to the single translation $T_{6,9}$.

Example 2. Find ST and TS if

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+3y \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x-y \\ -x+2y \end{bmatrix}.$$

Solution: By definition,

$$\begin{aligned}
(ST)\begin{bmatrix} x \\ y \end{bmatrix} &= S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) && \text{(Perform } T \text{ first.)} \\
&= S\begin{bmatrix} 4x-y \\ -x+2y \end{bmatrix} && \text{(Then perform } S\text{.)} \\
&= \begin{bmatrix} 2(4x-y)+(-x+2y) \\ (4x-y)+3(-x+2y) \end{bmatrix} \\
&= \begin{bmatrix} 7x \\ x+5y \end{bmatrix}.
\end{aligned}$$

Therefore $ST = R$, where

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x \\ x+5y \end{bmatrix}.$$

Also

$$\begin{aligned}(TS)\begin{bmatrix} x \\ y \end{bmatrix} &= T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= T\begin{bmatrix} 2x + y \\ x + 3y \end{bmatrix} \\ &= \begin{bmatrix} 4(2x + y) - (x + 3y) \\ -(2x + y) + 2(x + 3y) \end{bmatrix} \\ &= \begin{bmatrix} 7x + y \\ 5y \end{bmatrix}.\end{aligned}$$

Therefore $TS = U$, where

$$U\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x + y \\ 5y \end{bmatrix}.$$

Note that $TS \neq ST$ for these transformations S and T.

Example 3. Show that $R_{(\pi/4)}\, R_{(\pi/4)} = R_{(\pi/2)}$, where

$$R_{(\alpha)}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \alpha - y \sin \alpha \\ x \sin \alpha + y \cos \alpha \end{bmatrix}.$$

Solution:

$$x \cos \frac{\pi}{4} - y \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}\,(x - y)$$

$$x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}\,(x + y)$$

Therefore

$$\begin{aligned}(R_{(\pi/4)}\, R_{(\pi/4)})\begin{bmatrix} x \\ y \end{bmatrix} &= R_{(\pi/4)}\left(R_{(\pi/4)}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= R_{(\pi/4)}\begin{bmatrix} \frac{1}{\sqrt{2}}\,(x - y) \\ \frac{1}{\sqrt{2}}\,(x + y) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\,(x - y) - \frac{1}{\sqrt{2}}\,(x + y)\right\} \\ \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\,(x - y) + \frac{1}{\sqrt{2}}\,(x + y)\right\} \end{bmatrix} \\ &= \begin{bmatrix} -y \\ x \end{bmatrix}.\end{aligned}$$

Also

$$x \cos \frac{\pi}{2} - y \sin \frac{\pi}{2} = -y\,,$$

$$x \sin \frac{\pi}{2} + y \cos \frac{\pi}{2} = x\,.$$

Therefore

$$R_{(\pi/2)}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix},$$

and so

$$R_{(\pi/4)}\, R_{(\pi/4)} = R_{(\pi/2)}\,.$$

We may define the product $f \ominus g$ of two functions f and g to be the function $f(g)$. (We assume that the domain and range of f and g are such that $f(g)$ is indeed a function.) Again, $f \circ g$ is sometimes used to indicate this product.

Example 4. Find $f \ominus g$ and $g \ominus f$ if f and g are the functions

$$f : x \rightarrow 4x - 3, \qquad x \in Re\,,$$
$$g : x \rightarrow -2x + 5, \qquad x \in Re\,.$$

Solution:

$$f(g) \text{ is the function: } x \xrightarrow{g} -2x + 5$$
$$\xrightarrow{f} 4(-2x+5)-3\,,$$

that is, the function

$$f \ominus g : x \rightarrow -8x + 17, \qquad x \in Re\,.$$

$$g(f) \text{ is the function: } x \xrightarrow{f} 4x - 3$$
$$\xrightarrow{g} -2(4x-3) + 5\,,$$

that is, the function

$$g \ominus f : x \rightarrow -8x + 11, \qquad x \in Re\,.$$

Note that $f \ominus g \neq g \ominus f$ for this binary operation $\ominus$.

EXERCISE 8.7

1. Find the products, in both orders, of the following pairs of translations.
   (a) $T_{3,4}$ and $T_{1,1}$ (b) $T_{5,2}$ and $T_{3,-2}$
   (c) $T_{2,4}$ and $T_{-2,-4}$ (d) $T_{-1,-3}$ and $T_{-2,4}$
2. Show algebraically the truth of the following statements regarding rotations.
   (a) $R_{(\pi/2)}\, R_{(\pi/2)} = R_{(\pi)}$. (b) $R_{(\pi)}\, R_{(\pi)} = R_0$.
   (c) $R_{(\pi/6)}\, R_{(\pi/3)} = R_{(\pi/2)}$. (d) $R_{(\pi/3)}\, R_{(\pi/6)} = R_{(\pi/2)}$.
3. Find $f \ominus g$ and $g \ominus f$ (that is, $f(g)$ and $g(f)$) for the following functions defined on Re.
   (a) $f : x \rightarrow 2x + 7, \quad g : x \rightarrow 3x - 4$
   (b) $f : x \rightarrow -3x + 4, \quad g : x \rightarrow -6x + 1$
   (c) $f : x \rightarrow x^2 + 3, \quad g : x \rightarrow 3x - 2$
   (d) $f : x \rightarrow x^2 - x + 1, \quad g : x \rightarrow x^2 + x + 1$

8.8. Groupoids

We have been discussing binary operations Θ on the elements of a given set. We have seen that Θ acts on a pair of set elements a and b (in that order) to produce the element which we call $a \Theta b$. Now $a \Theta b$ may or may not be an element of the given set. If $a \Theta b$ is an element of the given set for every pair of elements a and b of the set, then we say that the set is closed under the binary operation Θ.

We define the following algebraic system.

DEFINITION. A groupoid is an algebraic system consisting of a set S and an operation Θ, which is such that S is closed under Θ.

A groupoid requires the following three things:

(i) a set S,

(ii) a binary operation Θ defined on the elements of S,

(iii) closure of S under Θ, that is,

$$a \Theta b \in S \quad \text{whenever } a, b \in S .$$

Example 1. Is the set of positive integers a groupoid under the binary operation Θ which is such that $a \Theta b = a - b$?

Solution: $3 \Theta 5 = 3 - 5 = -2$, which is not a positive integer. This algebraic system is not a groupoid.

Example 2. Is the system in Example (1) a groupoid if Θ is such that $a \Theta b = 3a + 4b$?

Solution: $3a + 4b$ is a positive integer for every pair (a, b) of positive integers; this system is a groupoid.

Example 3. Is the set of permutations on the symbols $\{A, B, C\}$ a groupoid under the binary operation Θ of multiplication of permutations defined in Section 8.6?

Solution: The product of two permutations on $\{A, B, C\}$ is always a permutation on the same set. Let P be the permutation

$$P = \begin{pmatrix} A & B & C \\ A* & B* & C* \end{pmatrix};$$

then $(A*, B*, C*)$ is simply a rearrangement of (A, B, C) so that we may write Q as

$$\begin{pmatrix} A* & B* & C* \\ A** & B** & C** \end{pmatrix},$$

and then

$$P \ominus Q = \begin{pmatrix} A & B & C \\ A* & B* & C* \end{pmatrix} \ominus \begin{pmatrix} A* & B* & C* \\ A** & B** & C** \end{pmatrix} = \begin{pmatrix} A & B & C \\ A** & B** & C** \end{pmatrix},$$

which is a permutation on $\{A, B, C\}$. For example, if

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

we may rewrite Q as

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix},$$

and then

$$P \ominus Q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \ominus \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

EXERCISE 8.8

Examine the following algebraic systems. (The set is given and the operation $\ominus$ is described or defined.) Which of the systems are groupoids?

1. (a) I; $\ominus$ is ordinary addition, $a \ominus b = a + b$.
   (b) Ra; $\ominus$ is ordinary multiplication, $a \ominus b = ab$.
   (c) Re; $\ominus$ is ordinary subtraction, $a \ominus b = a - b$.
2. (a) The set E of even integers under ordinary addition
   (b) The set E of even integers under ordinary multiplication
   (c) The set of positive even integers with $a \ominus b = a - b$
   (d) The set O of odd integers under ordinary addition
   (e) The set O of odd integers under ordinary multiplication
   (f) The set of positive odd integers under ordinary multiplication
3. (a) The set $\{0, 1\}$ under ordinary multiplication
   (b) The set $\{0, 1\}$ under ordinary addition
   (c) The set $\{-1, 0, 1\}$ under ordinary multiplication
4. (a) The set of all integral multiples of 5 under ordinary addition
   (b) The set of all integral multiples of 5 under ordinary multiplication
   (c) The set of all integral multiples of 5 with $a \ominus b = a - b$
   (d) The set of all integral multiples of 5 with $a \ominus b = a/b$
   (e) The set of all integral powers of 5 under ordinary multiplication
   (f) The set of all integral powers of 5 with $a \ominus b = a/b$

5. (a) The set of all real numbers of the form $a + b\sqrt{2}$, $a, b \in I$, under ordinary addition
   (b) The set of (a) under ordinary multiplication
   (c) The set of all real numbers of the form $a - b\sqrt{7}$, $a, b \in I$, under ordinary addition
   (d) The set of (c) under ordinary multiplication

6. (a) The set $\{1, -1, i, -i\}$ under multiplication of complex numbers (Construct a multiplication table.)
   (b) C; Θ is addition of complex numbers.
   (c) C; Θ is multiplication of complex numbers.
   (d) The set $\{1, (-1 + i\sqrt{3})/2, (-1 - i\sqrt{3})/2\}$ under multiplication of complex numbers (Construct a multiplication table.)
   (e) The set of all complex numbers $a + bi$ with $a^2 + b^2 = 1$ under multiplication of complex numbers
   (f) C; $a \Theta b = a/b$.

7. (a) N; $a \Theta b = 2a - b$.
   (b) N; $a \Theta b = a^2 + b^2$.
   (c) N; $a \Theta b = (a + b)^2$.
   (d) N; $a \Theta b = 2a + 3b$.
   (e) N; $a \Theta b = a/b$.
   (f) N; $a \Theta b =$ maximum of a and b.
   (g) N; $a \Theta b =$ l.c.m. of a and b.
   (h) N; $a \Theta b =$ g.c.d. of a and b.
   (i) N; $a \Theta b = a$.
   (j) N; $a \Theta b = b$.
   (k) N; $a \Theta b = (a + b)/2$.
   (l) N; $a \Theta b = a^b$.
   (m) N; $a \Theta b = a - ab$.
   (n) N; $a \Theta b = a + b + ab$.
   (o) N; $a \Theta b = \frac{1}{2}$ [(l.c.m. of a and b) + (g.c.d. of a and b)].
   (p) N; $a \Theta b =$ remainder when a is divided by b.
   (q) N; $a \Theta b =$ remainder when b is divided by a.

8. (a) The set $\{R_0, R_{180}, D_1, D_2\}$ of motions of the rhombus (Example (4), Section 8.6) under multiplication of mappings
   (b) $\{I, R_x, R_y, R_{xy}\}$ under multiplication of mappings (See question (6) Exercise 8.6.)
   (c) The set $\{R_0, R_{90}, R_{180}, R_{270}, H, V, D_1, D_2\}$ of motions of the square under multiplication of mappings. Set up a multiplication table. (The results of question (5), Exercise 8.6 will be useful.)
   (d) The subset $\{R_0, R_{180}, H, V\}$ of the set in (c) under multiplication of mappings
   (e) The set $\{R_0, R_{120}, R_{240}, E_1, E_2, E_3\}$ of motions of the equilateral triangle (see question (4), Exercise 8.6 for diagram) under multiplication of mappings. Construct a multiplication table.

9. (a) The set of 2×2 real matrices under addition of matrices
   (b) The set of 2×2 real matrices under multiplication of matrices

10. (a) The set of all real linear functions

$$f : x \rightarrow ax + b$$

with $f \Theta g = f(g)$

(b) The set of all real quadratic functions

$$f : x \rightarrow ax^2 + bx + c, \qquad a \neq 0,$$

with $f \Theta g = f(g)$

(c) The set in (a) with $f \Theta g$ defined as $f + g$

(d) The set in (b) with $f \Theta g$ defined as $f + g$

(e) The set of real functions

$$f_1 : t \rightarrow t \qquad f_2 : t \rightarrow \frac{1}{t} \qquad f_3 : t \rightarrow 1 - t$$

$$f_4 : t \rightarrow \frac{1}{1 - t} \qquad f_5 : t \rightarrow \frac{t - 1}{t} \qquad f_6 : t \rightarrow \frac{t}{t - 1}$$

with $f \Theta g = f(g)$. Construct a multiplication table.

11. (a) The set of translations of the plane under the multiplication of mappings

(b) The set of rotations of the plane under the multiplication of mappings

(c) The set $\{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\}$ of rotations of the plane under multiplication of mappings, noting that

$$R_{180} \Theta R_{240} = R_{420} = R_{60},$$

$$R_{180} \Theta R_{300} = R_{480} = R_{120},$$

etc.

12. (a) The set of all permutations on $\{A, B, C\}$ under the multiplication of permutations

(b) The set of all permutations on $\{1, 2, 3, 4\}$ under the multiplication of permutations

(c) The set of all permutations on $\{x_1, x_2, \cdots x_n\}$ under the multiplication of permutations

13. (a) The set $\{A, B, C\}$ with multiplication defined by the following table

Θ	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

(b) The set $\{A, B, C\}$ with multiplication defined by the table

θ	A	B	C
A	A	B	C
B	B	A	A
C	C	B	A

(c) The set $\{A, B, C\}$ with multiplication defined by the table

θ	A	B	C
A	A	B	C
B	A	B	C
C	A	B	C

(d) The set $\{A, B, C, D\}$ with multiplication defined by the table

θ	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

14. The set of transformations T such that

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}, \quad a, b, c, d \in Re,$$

under the multiplication of transformations.

15. The set of transformations T such that

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}, \quad a, b \in Re,$$

under the multiplication of transformations.

8.9. Semigroups

We have seen that a groupoid is an algebraic system consisting of a set S that is closed under a binary operation Θ. If, in addition, the operation Θ is associative, so that

$$(a \Theta b) \Theta c = a \Theta (b \Theta c) ,$$

for all a, b, and c in S, the algebraic system is called a semigroup.

DEFINITION. A semigroup is a groupoid in which the binary operation Θ is associative.

Thus a semigroup requires the following four things:

(i) a set S,

(ii) a binary operation Θ defined on the elements of S,

(iii) closure, $a \Theta b \in S$ whenever $a, b \in S$,

(iv) associativity, $(a \Theta b) \Theta c = a \Theta (b \Theta c)$ for all $a, b, c \in S$.

We will re-examine the examples of Section 8.8 to see if they involve semigroups.

Example 1. Is the set of positive integers a semigroup under the binary operation Θ such that $a \Theta b = a - b$?

Solution: This system is not a groupoid and hence not a semigroup.

Example 2. Is the system in Example 1 a semigroup if $a \Theta b = 3a + 4b$?

Solution: The system is a groupoid. Also,

$$\begin{aligned}(a \Theta b) \Theta c = (3a + 4b) \Theta c &= 3(3a + 4b) + 4c \\ &= 9a + 12b + 4c ,\end{aligned}$$

and

$$\begin{aligned}a \Theta (b \Theta c) = a \Theta (3b + 4c) &= 3a + 4(3b + 4c) \\ &= 3a + 12b + 16c .\end{aligned}$$

Therefore $(a \Theta b) \Theta c \neq a \Theta (b \Theta c)$ and the system is not a semigroup.

Example 3. Is the set of permutations on the symbols $\{A, B, C\}$ a semigroup under the multiplication of permutations?

Solution: We showed in Example 3, Section 8.8 that this set is a groupoid under the multiplication of permutations. Let P, Q, and R be permutations on these symbols and let

$$P = \begin{pmatrix} A & B & C \\ A^* & B^* & C^* \end{pmatrix} .$$

Then we may write

$$Q = \begin{pmatrix} A^* & B^* & C^* \\ A^{**} & B^{**} & C^{**} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} A^{**} & B^{**} & C^{**} \\ A^{***} & B^{***} & C^{***} \end{pmatrix},$$

where (A*, B*, C*) and (A**, B**, C**) are rearrangements of (A, B, C). Then

$$\begin{aligned}(P \ominus Q) \ominus R &= \left\{\begin{pmatrix} A & B & C \\ A^* & B^* & C^* \end{pmatrix} \ominus \begin{pmatrix} A^* & B^* & C^* \\ A^{**} & B^{**} & C^{**} \end{pmatrix}\right\} \ominus R \\ &= \begin{pmatrix} A & B & C \\ A^{**} & B^{**} & C^{**} \end{pmatrix} \ominus \begin{pmatrix} A^{**} & B^{**} & C^{**} \\ A^{***} & B^{***} & C^{***} \end{pmatrix} \\ &= \begin{pmatrix} A & B & C \\ A^{***} & B^{***} & C^{***} \end{pmatrix}.\end{aligned}$$

The reader is requested to show that it is also true that

$$P \ominus (Q \ominus R) = \begin{pmatrix} A & B & C \\ A^{***} & B^{***} & C^{***} \end{pmatrix}.$$

Therefore

$$(P \ominus Q) \ominus R = P \ominus (Q \ominus R)$$

for any three such permutations and so the system is a semigroup.

Example 4. Does the set of motions $\{R_0, R_{180}, D_1, D_2\}$ of the rhombus, described in Example 4, Section 8.6, form a semigroup under the multiplication of mappings?

Solution: That the system is a groupoid is shown by examining the multiplication table (why?). To check that it is a semigroup, we would have to check that

$$(P \ominus Q) \ominus R = P \ominus (Q \ominus R)$$

for any three motions P, Q, and R. This would mean checking 4 × 4 × 4 such cases and this is tedious. Instead we use the following general statement.

General Statement. If a set of *mappings* is closed under multiplication of mappings, then the multiplication is associative.

The reasoning here is that $(P \ominus Q) \ominus R$ and $P \ominus (Q \ominus R)$ both imply the mapping equivalent to performing mappings P, Q, and R in that order, assuming that the products involved can be formed (this is possible if we have closure). Thus the system in Example 4 is a semigroup since we have already shown closure by means of a multiplication table.

EXERCISE 8.9

Check to see which of the groupoids in Exercise 8.8 are semigroups.

8.10. Identity or Neutral Elements

We know that the sets *Ra*, *Re*, and *C* contain certain special elements called identity or neutral elements; 0 is the identity element under the operation of addition and 1 is the identity element under multiplication.

In general, an identity or neutral element of an algebraic system consisting of a set S with a binary operation Θ is an element e such that

$$e \,\Theta\, a = a \,\Theta\, e = a$$

for all $a \in S$.

We will see in the examples and exercises that it may be possible to have

$$e \,\Theta\, a = a$$

for all $a \in S$, but

$$a \,\Theta\, e \neq a\,.$$

In this case, e is called a left identity of the system. On the other hand, if

$$a \,\Theta\, e = a$$

for all $a \in S$, we call e a right identity of the system. If e is both a left and a right identity, we call it simply an identity.

Example 1. Does the set of integers under the operation Θ such that $a \,\Theta\, b = a - b$ possess an identity element?

Solution:

$$a \,\Theta\, 0 = a$$

for all $a \in I$, but

$$0 \,\Theta\, b = -b\,.$$

Thus the system possesses a right identity 0 but no left identity.

Example 2. Does the set of integers under the operation Θ such that $a \,\Theta\, b = 3a + 4b$ possess an identity element?

Solution: Suppose that $4 \,\Theta\, b = 4$ for some element b (the element b is then a right identity *for the element* 4); then

$$4 \,\Theta\, b = 12 + 4b = 4\,,$$

and so

$$b = -2\,.$$

But

$$3 \,\Theta\, (-2) = 3(3) + 4(-2) = 1 \neq 3$$

and so -2 is not a right identity for 3. Thus there is no right identity in this system and similarly it can be shown that there is no left identity.

We have already seen that the set of permutations on $\{A, B, C\}$ contains an identity or neutral permutation

$$I = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix},$$

and that the sets of motions of various plane figures that we have studied all possess an identity or neutral motion R_0, a counterclockwise rotation of 0° about the centre of the figure.

We have seen that some algebraic systems possess identity or neutral elements, some possess right identities but not left identities (Example 1), or left identities but not right, and some possess no identities at all (Example 2). The following theorem tells us that an algebraic system can have at most one identity element.

Theorem. If e and f are identity elements in an algebraic system, then

$$e = f.$$

Proof: If e is an identity, then

$$e \Theta a = a \Theta e = a$$

for all a.

If f is an identity, then

$$f \Theta a = a \Theta f = a$$

for all a. Therefore

$$e \Theta f = f,$$

since e is an identity, and

$$e \Theta f = e,$$

since f is an identity. Therefore

$$e = f, \qquad \text{as required.}$$

EXERCISE 8.10

1. Check to see which of the algebraic systems in Exercise 8.8
   (a) have no identities,
   (b) have right but no left identities,
   (c) have left but no right identities,
   (d) have both left and right identities.
2. Prove that if e is a *left* identity in an algebraic system and f is a *right* identity, then $e = f$.

8.11. Inverses

We know that, to every element of the sets Ra, Re, and C, there corresponds in the same set an element called its additive inverse. Thus $-3/2$ is the additive inverse of $3/2$ in Ra, $-\sqrt{2}$ is the additive inverse of $\sqrt{2}$ in Re, and $-3 - 2i$ is the additive inverse of $3 + 2i$ in C.

Similarly, to every *nonzero* element of the sets Ra, Re, and C, there corresponds in the same set an element called its multiplicative inverse. Thus $2/3$ is the multiplicative inverse of $3/2$ in Ra, $1/\sqrt{2}$ is the multiplicative inverse of $\sqrt{2}$ in Re, and $1/(3 + 2i)$ or $3/13 - 2i/13$ is the multiplicative inverse of $3 + 2i$ in C.

In general, if an algebraic system consisting of a set S and a binary operation Θ possesses an identity element e (and hence a unique identity), then an element $a \in S$ is said to possess a multiplicative inverse b if $b \in S$ and

$$a \Theta b = b \Theta a = e .$$

Of course it may be possible in some systems that $a \Theta b = e$ but $b \Theta a \neq e$; in this case b is called a right inverse of a. (Similarly left inverses may exist.) Also some elements of a system may have an inverse even if others do not.

Example 1. Does every element of the algebraic system consisting of the set of positive rationals under ordinary multiplication possess an inverse?

Solution: The system contains a multiplicative identity 1 and, for every positive rational a/b, we know that b/a is a positive rational such that

$$\frac{a}{b} \times \frac{b}{a} = \frac{b}{a} \times \frac{a}{b} = 1 .$$

Every element of this system possesses a multiplicative inverse.

Example 2. Does every element of the system with set Ra and ordinary multiplication possess an inverse?

Solution: The rational number 0 does not possess an inverse in this system; the other elements do.

Example 3. Does every element of the system with set Ra and ordinary addition possess an inverse?

Solution: This system contains an identity 0 and the inverse of the rational number a/b is the rational number $-a/b$. Every element of this system possesses an inverse.

Example 4. What is the inverse of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} ?$$

Solution: The inverse of P is the permutation

$$Q = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

found by inverting the symbol for P. We may write

$$Q = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

We easily check that

$$P \ominus Q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \ominus \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

and

$$Q \ominus P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \ominus \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I.$$

Example 5. Examine for inverses the system of permutations on $\{A, B, C, D\}$ under the multiplication of permutations.

Solution: The system possesses an identity permutation

$$I = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}.$$

Also, corresponding to every permutation

$$P = \begin{pmatrix} A & B & C & D \\ A^* & B^* & C^* & D^* \end{pmatrix},$$

there is a permutation

$$Q = \begin{pmatrix} A^* & B^* & C^* & D^* \\ A & B & C & D \end{pmatrix}$$

such that

$$P \ominus Q = Q \ominus P = I.$$

Example 6. Examine for inverses the system consisting of the set $\{1, 2, 3, 4\}$ with multiplication $\ominus$ defined by the table

$\ominus$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Solution: The system possesses an identity element 1. Also, from the table

$$\begin{aligned} 1 \ominus 1 &= 1\,, \\ 2 \ominus 3 = 3 \ominus 2 &= 1\,, \\ 4 \ominus 4 &= 1\,, \end{aligned}$$

so that every element of the system possesses an inverse. The elements 1 and 4 are their own inverses; 2 and 3 are the inverses of each other.

Example 7. State the inverse of the elements R_0, R_{180}, D_1, D_2 of the system in Example 4, Section 8.6.

Solution: From the multiplication table we see that the identity is R_0 and that

$$R_0 \ominus R_0 = R_0; \quad R_{180} \ominus R_{180} = R_0; \quad D_1 \ominus D_1 = R_0; \quad D_2 \ominus D_2 = R_0\,.$$

Thus each element is its own inverse.

The following theorem shows that, if an element a in a semigroup possesses an inverse, that inverse is unique.

Theorem. If a is an element of a semigroup with an identity e and

$$a \ominus b = b \ominus a = e$$

and also

$$a \ominus c = c \ominus a = e\,,$$

then

$$b = c\,.$$

Proof:

$$\begin{aligned} b &= b \ominus e && \text{(e is the identity.)} \\ &= b \ominus (a \ominus c) && \text{(given)} \\ &= (b \ominus a) \ominus c && \text{(associativity in semigroups)} \\ &= e \ominus c && \text{(given)} \\ &= c && \text{(identity).} \end{aligned}$$

Therefore

$$b = c,$$

as required.

If an element a of an algebraic system with identity e possesses a unique inverse, we use the symbol a^{-1} to represent that element; thus

$$a \ominus a^{-1} = a^{-1} \ominus a = e\,.$$

Then, for example, we refer to the inverse of the permutation P as P^{-1}; if

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

then (see Example 4)

$$P^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Example 8. Find the inverse of the function

$$f : x \to 2x - 3\,, \qquad x \in Re$$

in the algebraic system of real linear functions if $f \ominus g = f(g)$.

Solution: The function

$$f_1 : x \to x\,, \qquad x \in Re\,,$$

is the identity function in this algebraic system. Then the function

$$g : x \to ax + b\,, \qquad x \in Re$$

is an inverse for f if

$$f(g) = g(f) = f_1\,.$$

Now

$$\begin{aligned} f(g) : x \to\, & 2(ax + b) - 3 \\ = & \; 2ax + (2b - 3) \end{aligned}$$

and

$$2ax + (2b - 3) = x$$

if and only if

$$2a = 1 \qquad \text{and} \qquad 2b - 3 = 0\,.$$

Then g is the function,

$$g : x \to \frac{1}{2}x + \frac{3}{2}, \qquad x \in Re\,.$$

Check:

$$\begin{aligned} g(f) : x \to & \frac{1}{2}\,(2x - 3) + \frac{3}{2} \\ = & \; x\,. \end{aligned}$$

We use the symbol f^{-1} for this inverse function.

$$f^{-1} : x \to \frac{1}{2}x + \frac{3}{2}, \qquad x \in Re.$$

EXERCISE 8.11

1. Check those algebraic systems of Exercise 8.8 that contain an identity (omit 9(b)) to see which systems contain inverses for all elements. List the inverses for all elements in systems containing a finite number of elements.

2. Write the inverses of the following permutations.

   (a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$

   (d) $\begin{pmatrix} A & B & C & D & E \\ C & A & D & E & B \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 2 & 4 & 3 \end{pmatrix}$

3. Find the inverses of the following functions.

   (a) $f : x \rightarrow 4x - 1, \quad x \in Re$ (b) $f : x \rightarrow \frac{1}{2}x + 3, \quad x \in Re$

   (c) $f : x \rightarrow -2x + 3, \quad x \in Re$ (d) $f : x \rightarrow -4x - 7, \quad x \in Re$

4. Prove that if a is an element of a semigroup with identity e and if $a \ominus b = e$ and $c \ominus a = e$, then $b = c$.

5. Prove that if every element of a semigroup with identity e has a left inverse, then every element also has a right inverse.

8.12. Groups

If a semigroup possesses an identity element and every element possesses an inverse, we call the algebraic system a group.

DEFINITION. A group is a semigroup with an identity element, in which each element possesses an inverse.

We may restate this definition as follows.

DEFINITION. A group is a set of elements G together with a binary operation $\ominus$ such that

(1) G is closed under $\ominus$;

(2) $(a \ominus b) \ominus c = a \ominus (b \ominus c)$ for all $a, b, c \in G$;

(3) $e \in G$ such that $e \ominus a = a \ominus e = a$ for $a \in G$;

(4) for every $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1} \ominus a = a \ominus a^{-1} = e$.

We refer to the binary operation $\ominus$ as group multiplication. If, in addition, the elements of a group G satisfy the postulate

(5) $a \ominus b = b \ominus a$ for all $a, b \in G$,

then we call G a commutative or abelian group (named after the brilliant Norwegian mathematician Niels Henrik Abel (1802-1829)). Otherwise G is a noncommutative or nonabelian group.

Example 1. Show that the set Ra^+ is a group under ordinary multiplication.

Solution:

(1) Ra^+ is closed under multiplication.

(2) $(ab)c = a(bc)$ for $a, b, c \in Ra^+$.

(3) $1 \times a = a \times 1 = a$ for all $a \in Ra^+$.

(4) Every positive rational a/b has an inverse b/a which is a positive rational.

Thus this system is a group. In addition, $ab = ba$ for any two positive rationals a and b; hence this system is a commutative group.

Example 2. Show that the set Ra is not a group under ordinary multiplication.

Solution: The rational number 0 does not possess a multiplicative inverse.

Example 3. Show that the set of nonzero integers is not a group under ordinary multiplication.

Solution: The integer 7 does not possess an inverse in this set. (1/7 is not a member of this set.)

Example 4. Is the set Re a group under ordinary addition?

Solution:

(1) Re is closed under addition.

(2) $(a + b) + c = a + (b + c)$ for $a, b, c \in Re$.

(3) $0 + a = a + 0 = a$ for all $a \in Re$.

(4) $(-a) + a = a + (-a) = 0$ for all $a \in Re$.

This system is a group. In addition, this system is an abelian group (why?).

Example 5. Show that the set of permutations on {A, B, C} is a group under the multiplication of permutations.

Solution: We have already shown closure, associativity, and the existence of an identity and inverses; this system is a group. We have also seen that

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{C} & \mathrm{A} & \mathrm{B} \end{pmatrix} \theta \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} \end{pmatrix} = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{B} & \mathrm{A} & \mathrm{C} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} \end{pmatrix} \theta \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{C} & \mathrm{A} & \mathrm{B} \end{pmatrix} = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{C} & \mathrm{B} & \mathrm{A} \end{pmatrix}$$

so that this group is a noncommutative group.

Example 6. Is the system consisting of the set $\{1, 2, 3, 4\}$ with multiplication θ defined by the following table a group?

θ	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Solution: We see from the table that the set is closed under multiplication and that the set contains the identity 1. We have seen earlier that each element possesses an inverse.

$$1 \theta 1 = 1; \quad 2 \theta 3 = 3 \theta 2 = 1; \quad 4 \theta 4 = 1.$$

Finally we may check that associativity holds. Similarly, we may check that commutativity holds so that the system is a commutative group.

Example 7. Show that the set $\{R_0, R_{180}, D_1, D_2\}$ of motions of the rhombus (Example 4, Section 8.6) is a group under multiplication of mappings.

Solution: By examining the multiplication table

θ	R_0	R_{180}	D_1	D_2
R_0	R_0	R_{180}	D_1	D_2
R_{180}	R_{180}	R_0	D_2	D_1
D_1	D_1	D_2	R_0	R_{180}
D_2	D_2	D_1	R_{180}	R_0

we may check that all of the postulates for an abelian group hold.

Example 8. Show that the set of real linear functions

$$f : x \rightarrow ax + b, \qquad a \neq 0, \qquad x \in Re$$

is a group under the binary operation Θ with $f \Theta g = f(g)$.

Solution:

(1) If f and g are the functions

$$\begin{aligned} f &: x \rightarrow ax + b, \qquad a \neq 0, \qquad x \in Re, \\ g &: x \rightarrow cx + d, \qquad c \neq 0, \qquad x \in Re, \end{aligned}$$

then

$$\begin{aligned} f(g) : x &\rightarrow a(cx + d) + b \\ &= (ac)x + (ad + b). \end{aligned}$$

Thus the set of such functions is closed under Θ.

(2) Θ is associative. (See question (5), Exercise 8.12.)

(3) $f_1 : x \rightarrow x$, $x \in Re$, is the identity.

(4) $f^{-1} : x \rightarrow \frac{1}{a}x - \frac{b}{a}$, $x \in Re$, is the inverse of f. (See question (6), Exercise 8.12.)

The system is a group but it is not a commutative group. (See question (7), Exercise 8.12.)

Will the result of this example be true for the set of all linear functions

$$f : x \rightarrow ax + b, \qquad a \neq 0,$$

if a and b are integers? rational numbers?

Some groups contain a finite number of elements and we call such groups finite groups; if a group contains n elements, it is called a group of order n. Thus the group of Example 5 has order 6, while those of Examples 6 and 7 both have order 4. If a group is not a finite group, then it is an infinite group; the groups of Examples 1, 4, and 8 are infinite groups.

We will see in the following exercises that the sets of motions associated with certain geometrical figures form groups. This is true in general for such geometrical figures. As we have seen, the number of such motions depends on the amount of symmetry of the geometrical figure. Thus the order of the group associated in this way with a geometrical figure will be related to the symmetry of the figure: the greater the order of the group, the greater the symmetry of the corresponding figure.

EXERCISE 8.12

1. Re-examine the algebraic systems of Exercise 8.8 (omit 9(b)) to determine which of them are groups, making use of the results of Exercises 8.8 to 8.11.
2. Which of the above groups are commutative groups?
3. Under which of the following definitions of multiplication will the set $\{1, 2, 3, 4\}$ form a group? Give a complete discussion in each case.

(a)

Θ	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	2	1
4	4	3	1	2

(b)

Θ	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

(c)

Θ	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

(d)

Θ	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

4. Which quadrilateral other than a square has the most symmetry? Justify your answer in terms of the order of the corresponding group of motions.
5. Prove that the binary operation Θ of Example 8 is associative.
6. Verify that the function labelled f^{-1} in Example 8 is indeed the inverse of f.
7. Prove that the group in Example 8 is not a commutative group.
8. Prove that if a and b are elements of a group G, then the linear equation $a \Theta x = b$ has a unique solution in G.
9. Repeat question (8) for the linear equation $y \Theta a = b$.
10. Why is it necessary to consider the two types of linear equations in questions (8) and (9)?

8.13. Groups of Matrices Under Multiplication

Consider the set of all real 2 × 2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We have defined the product of two such matrices to be

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}.$$

Example 1. Find the product of the matrices

$$A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}.$$

Solution:

$$A\,B = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} (3)(4) + (5)(2) & (3)(-3) + (5)(7) \\ (-1)(4) + (2)(2) & (-1)(-3) + (2)(7) \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 26 \\ 0 & 17 \end{bmatrix}.$$

It is immediate from the definition that the set of all 2 × 2 real matrices is closed under multiplication. It is also true that multiplication is associative; this is verified (but not proved!) in the following example.

Example 2. Find $(AB)C$ and $A(BC)$ if A and B are as in Example 1 and

$$C = \begin{bmatrix} 2 & 1 \\ -5 & 3 \end{bmatrix}.$$

Solution: From Example 1,

$$(A\,B)\,C = \begin{bmatrix} 22 & 26 \\ 0 & 17 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -86 & 100 \\ -85 & 51 \end{bmatrix}.$$

Also

$$B\,C = \begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 23 & -5 \\ -31 & 23 \end{bmatrix}$$

so that

$$A\,(B\,C) = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 23 & -5 \\ -31 & 23 \end{bmatrix} = \begin{bmatrix} -86 & 100 \\ -85 & 51 \end{bmatrix}.$$

Therefore $(A\,B)\,C = A\,(B\,C)$ for these matrices.

Now the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is an identity in this system. (See question (4), Exercise 8.13.)

Finally, the matrix

$$\begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

is an inverse for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

provided $ad - bc \neq 0$. (See question (5), Exercise 8.13.)

Combining these results we conclude that the set of all real 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $ad - bc \neq 0$ is a group under the multiplication of matrices.

The following example shows that this group is not a commutative group.

Example 3. Evaluate BA for A and B as in Example 1.

Solution:

$$BA = \begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 14 \\ -1 & 24 \end{bmatrix}.$$

Therefore $AB \neq BA$ since 2×2 real matrices are equal if and only if corresponding entries are equal.

EXERCISE 8.13

1. Find AB and BA for the following matrices.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$. (b) $A = \begin{bmatrix} -1 & 5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix}$. (d) $A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 5 & -2 \end{bmatrix}$.

2. Find $(AB)C$ and $A(BC)$ for the matrices A and B of question (1) and the matrix

$$C = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}.$$

3. Find the inverses A^{-1} and B^{-1} of the matrices A and B of question (1). Check your answers by multiplication.

4. Prove that $IA = AI = A$ for every real 2×2 matrix if

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5. Show that the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with } ad - bc \neq 0$$

is

$$\begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

6. Prove that the set of all real 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with $ab \neq 0$ is a commutative group.

7. Is the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with a, b, c, d integers and $ad - bc \neq 0$, a group under multiplication? If not, which of the group postulates are violated?

8. The rotation of the plane $(x, y) \to (u, v)$ with

$$u = x \cos \alpha - y \sin \alpha$$
$$v = x \sin \alpha + y \cos \alpha$$

may be represented by the matrix

$$R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

Show that $R_\alpha R_\beta = R_{\alpha + \beta} = R_\beta R_\alpha$. Explain the geometric significance of this.

9. Show that $R_\alpha^{-1} = R_{(-\alpha)}$ in question (8).

10. Prove that the set of matrices R_α is a group.

11. The reflections in the plane given by the following systems of equations

$$I\begin{cases} u = x + 0y \\ v = 0x + y \end{cases} \qquad R_x\begin{cases} u = x + 0y \\ v = 0x - y \end{cases}$$

$$R_y\begin{cases} u = -x + 0y \\ v = 0x + y \end{cases} \qquad R_{xy}\begin{cases} u = -x + 0y \\ v = 0x - y \end{cases}$$

can be represented by the following matrices. (We use the same symbols for the matrices.)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_{xy} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show that the set $\{I, R_x, R_y, R_{xy}\}$ of four matrices is a group under multiplication. Construct a multiplication table for this group.

8.14. Rings

We have now gone as far as we can go in this course in discussing algebraic systems with a single binary operation. In this section and the next we introduce two mathematical systems that contain *two* binary operations. The first of these is the algebraic system known as a ring.

A ring is an algebraic system consisting of a set of elements and two binary operations with the system obeying certain postulates. We call the two operations "addition" and "multiplication", and we use the symbols $a \oplus b$ and $a \otimes b$ or ab for their result on two elements a and b, even though these operations may bear absolutely no similarity to ordinary addition and multiplication.

DEFINITION. A ring is an algebraic system consisting of a set R of elements and two binary operations $\oplus$ and $\otimes$ such that

(1) R is a commutative group under $\oplus$;

(2) R is closed under $\otimes$;

(3) multiplication is associative:

$$(ab)c = a(bc) \text{ or } (a \otimes b) \otimes c = a \otimes (b \otimes c);$$

(4) multiplication is distributive over addition:

$$a(b \oplus c) = ab \oplus ac \text{ and } (a \oplus b)c = ac \oplus bc.$$

Example 1. Show that the set I of integers is a ring under ordinary addition and multiplication.

Solution:

(1) We have already checked that I is a commutative group under addition (Exercise 8.12).

(2) I is closed under multiplication.

(3) Multiplication of integers is associative.

(4) Multiplication is distributive over addition.

Thus I is a ring under $+$ and $\times$. In addition, this algebraic system contains a multiplicative identity; we say that this system is a ring with unit or a ring with multiplicative identity. Further, the multiplication is commutative; the system is thus a commutative ring with unit.

Example 2. Is the set I^+ of positive integers a ring under ordinary addition and multiplication?

Solution: I^+ is not a group under addition since I^+ does not contain 0 or additive inverses. Thus I^+ cannot be a ring under $+$ and $\times$.

Example 3. Show that the set of all integral multiples of 3 is a ring under ordinary addition and multiplication.

Solution:

(1) (a) The sum of two multiples of 3 is a multiple of 3.
  (b) Addition of integers is associative.
  (c) 0 is a multiple of 3.
  (d) $3m$ and $-3m$ are both multiples of 3 if m is an integer.
  (e) Addition of integers is commutative.

Therefore this set is a commutative group under $+$.

(2) The product of two multiples of 3 is a multiple of 3.

(3) Multiplication of integers is associative.

(4) Multiplication of integers is distributive over addition.

Thus the set of all integral multiples of 3 is a ring under $+$ and $\times$. The system does not contain a multiplicative identity since 1 is not an integral multiple of 3. However, multiplication of integers is commutative so that this system is a commutative ring.

Example 4. Show that the set $\{0, 1\}$ is a ring under addition and multiplication as defined by the tables .

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

Solution:

(1) By examining the table for addition, we see that $\{0, 1\}$ is a commutative group under addition. (Check the postulates.)

(2) The table for $\otimes$ shows closure under this operation.

(3) Associativity may be checked from the table.

(4) To check distributivity, we need only observe that

$$1 \otimes (0 \oplus 0) = 1 \otimes 0 = 0 = 1 \otimes 0 \oplus 1 \otimes 0,$$
$$1 \otimes (1 \oplus 0) = 1 \otimes 1 = 1 = 1 \otimes 1 \oplus 1 \otimes 0,$$
$$1 \otimes (1 \oplus 1) = 1 \otimes 0 = 0 = 1 \otimes 1 \oplus 1 \otimes 1,$$

since multiplication by 0 is obviously distributive. Therefore this system is a ring. Check that it is a commutative ring with unit.

EXERCISE 8.14

1. Check to see which of the following algebraic systems are rings under the operations of ordinary addition and multiplication.
   - (a) The set of even integers
   - (b) The set of integral multiples of 5
   - (c) The set of rational numbers
   - (d) The set of positive rationals
   - (e) The set of rationals of the form $a/2$, where a is an integer
   - (f) The set of rationals of the form $a/7$, where a is an integer
   - (g) The set of real numbers
   - (h) The set of positive real numbers
   - (i) The set of real numbers of the form $a + b\sqrt{2}$, a and b integers
   - (j) The set $\{1, -1, i, -i\}$ of complex numbers
   - (k) The set of all complex numbers
   - (l) The set of all complex numbers $a + bi$ with a and b even integers.
   - (m) The set of all complex numbers $a + bi$ with a and b rational

2. Is the set of all polynomials with integral coefficients a ring under the addition and multiplication of polynomials?

3. Is the set of all real 2×2 matrices a ring under the addition and multiplication of matrices?

4. (a) Is the set of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in I,$$

   a ring under the addition and multiplication of matrices?
   - (b) Is it a ring if $b = c = 0$?
   - (c) Is it a ring if $b = c = 0$ and $d = a$?
   - (d) Is it a ring if $b = c = 0$ and $d = 2a$?
   - (e) Is it a ring if $b = c = 0$ and $d = a^2$?

5. Is the set $\{0, 1, 2\}$ with addition and multiplication defined by the following tables, a ring?

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

6. Which of the rings in questions (1) to (5) are
   (a) commutative rings?
   (b) rings with unit?
   (c) commutative rings with unit?

7. In which of the systems in questions (1) to (5) is it possible to have $ab = 0$ and yet $a \neq 0$ and $b \neq 0$?

8. Why is it necessary to stipulate both distributive laws
$$a(b \oplus c) = ab \oplus ac \text{ and } (a \oplus b)c = ac \oplus bc$$
in the postulates for a ring?

9. A ring with a finite number of elements is called a finite ring; otherwise it is an infinite ring. Which of the examples of this section are finite rings? Which of the rings in questions (1) to (5) are finite rings?

8.15. Fields

Finally we consider the postulates of the algebraic system called a field. We have already considered these postulates at other stages in our study of the rational, real, and complex number systems, but we present them here formally in order to round out the present discussion.

DEFINITION. A field is an algebraic system consisting of a set F and two binary operations called addition and multiplication and such that

(1) F is a commutative group under addition,

(2) $F - 0$ (the set of nonzero elements of F) is a commutative group under multiplication,

(3) multiplication is distributive over addition.

We list the eleven field postulates implied in questions (1) to (3) in full. A field is an algebraic system consisting of a set F and two binary operations called addition and multiplication and such that

(i) F is closed under addition,
(ii) addition is associative,
(iii) F contains an identity 0 under addition,
(iv) F contains an additive inverse $(-a)$ for each element a,

(v) addition is commutative,

(vi) F is closed under multiplication,

(vii) multiplication is associative,

(viii) F contains an identity 1 under multiplication,

(ix) F contains a multiplicative inverse a^{-1} for each nonzero element a,

(x) multiplication is commutative,

(xi) multiplication is distributive over addition.

We have stated that the sets of rational numbers, real numbers, and complex numbers are fields under addition and multiplication. Indeed it was through a study of the properties of these systems that the definition of a field was formulated.

Example 1. Is the set I of integers a field under ordinary + and ×?

Solution: All of the postulates for a field are satisfied by I under + and ×, except the existence of multiplicative inverses for nonzero elements; there is no integer which is the multiplicative inverse of 7 for example. Therefore this system is not a field.

Example 2. Is the set of all integral multiples of 3 a field under + and ×?

Solution: We have seen (Example 3, Section 8.14) that this system is a commutative ring. However it again does not contain multiplicative inverses and so is not a field.

Example 3. Show that the set $\{0, 1\}$ of integers is a field under addition and multiplication as defined by the tables

⊕	0	1
0	0	1
1	1	0

⊗	0	1
0	0	0
1	0	1

Solution: We have seen (Example 4, Section 8.14) that this system is a commutative ring with unit. The only field postulate we need to check is the existence of inverses for nonzero elements. But the only nonzero element is 1 and it is its own multiplicative inverse. Hence this system is a field; it is an example of a finite field.

EXERCISE 8.15

1. Check to see which of the algebraic systems of Exercise 8.14 are fields.
2. Is the set $\{0, 1, 2, 3\}$ a field under the following operations?

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

3. Show that if a and b belong to a field, then ab can equal zero if and only if at least one of a or b is zero.
4. Is the set of all real 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$ a field under addition and multiplication of matrices?

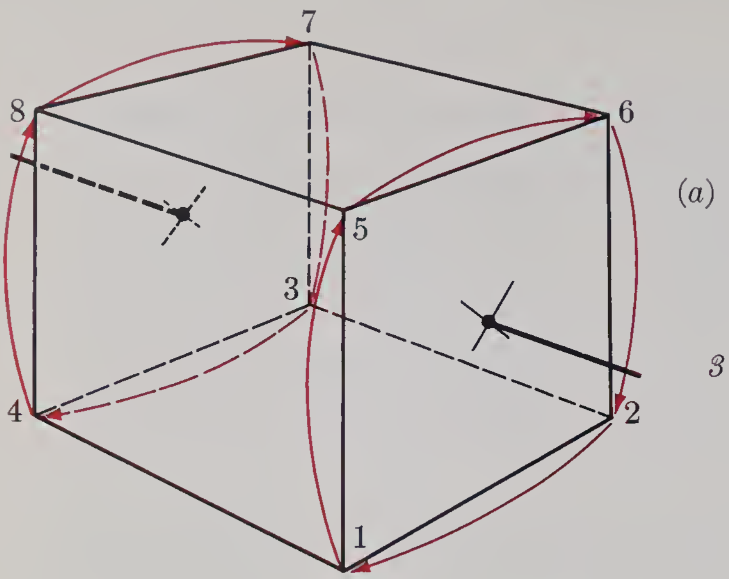
8.16. Rigid Motions of the Cube and Regular Tetrahedron (Supplementary)

In this section we will consider the sets of rigid rotations that map the cube into itself and those that map the regular tetrahedron into itself.

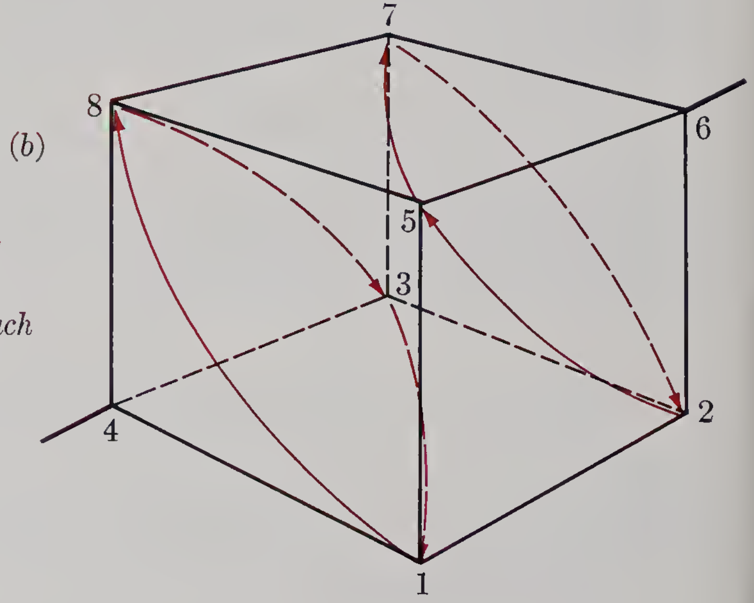
It will prove very helpful if the student constructs a cardboard model of the cube and the regular tetrahedron. (The latter can be constructed by taping together four equilateral triangles.) By a rigid motion of the cube, we will mean a motion that can actually be performed in space; we will not consider any other mappings at this time. (Recall that in studying the motions of a square we considered reflections along a diagonal but pointed out that the same result could be obtained by a rotation of 180° (in space) about that diagonal.)

Rigid Motions of the Cube

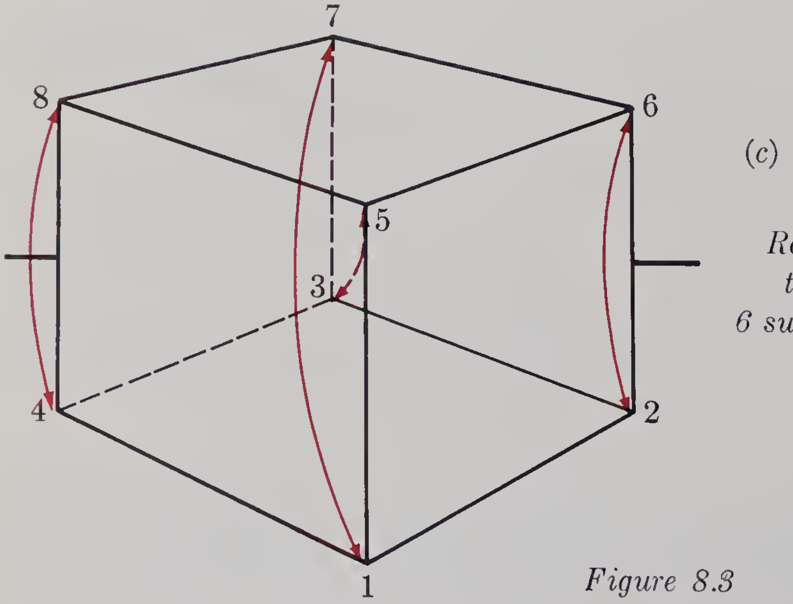
The three types of rigid motions of the cube are exhibited in Figure 8.3. The three types are the following.



Rotation about the line joining
the centres of opposite faces
3 such lines, 3 rotations about each
—9 rotations



Rotation about the line joining
opposite vertices
4 such lines, 2 rotations about each
—8 rotations



Rotation about the line joining
the centres of opposite edges
6 such lines, 1 rotation about each
—6 rotations

Figure 8.3

Type 1: rotation about a line joining the centres of opposite faces,

Type 2: rotation about a line joining opposite vertices,

Type 3: rotation about a line joining the centres of opposite edges.

Type 1: Figure 8.3 (*a*) indicates a rotation of 90° about the line joining the centres of faces 1 2 6 5 and 4 3 7 8. Rotations of 90°, 180°, and 270° about this line are possible (rotation of 360° or 0° is the identity rotation). The rotation of 90° about this line produces the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 4 & 8 & 6 & 2 & 3 & 7 \end{pmatrix}$$

on the vertices of the cube. (Write the permutations corresponding to rotations of 180° and 270°.)

Since there are three such lines possible, there are 3×3 or 9 rigid motions of Type 1 of the cube (apart from the identity).

Type 2: Figure 8.3 (*b*) indicates a rotation of 120° about the line joining opposite vertices 4 and 6. Rotations of 120° and 240° about this line are possible. Rotation of 360° or 0° is the identity. The rotation of 120° shown produces the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 1 & 4 & 7 & 6 & 2 & 3 \end{pmatrix}$$

on the vertices of the cube. Note that 4 and 6 are unchanged. (Write the permutation corresponding to a rotation of 240°.)

Since there are four such lines possible, there are 4×2 or 8 rigid motions of Type 2 of the cube (apart from the identity).

Type 3: Figure 8.3 (*c*) indicates a rotation of 180° about the line joining the midpoints of the edges 2 6 and 4 8; this rotation produces the permutation

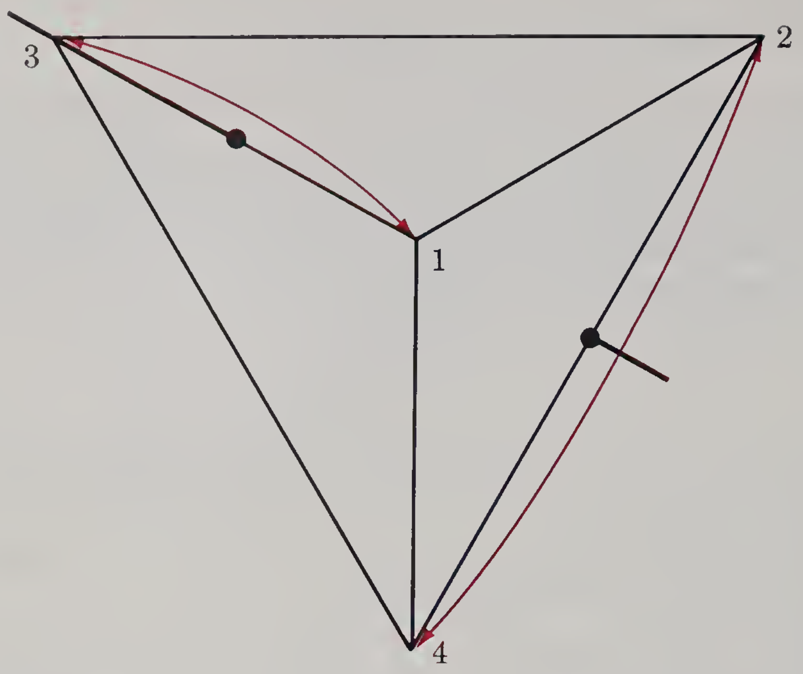
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 8 & 3 & 2 & 1 & 4 \end{pmatrix}$$

There are six such lines and so there are six rigid motions of Type 3 for the cube (apart from the identity).

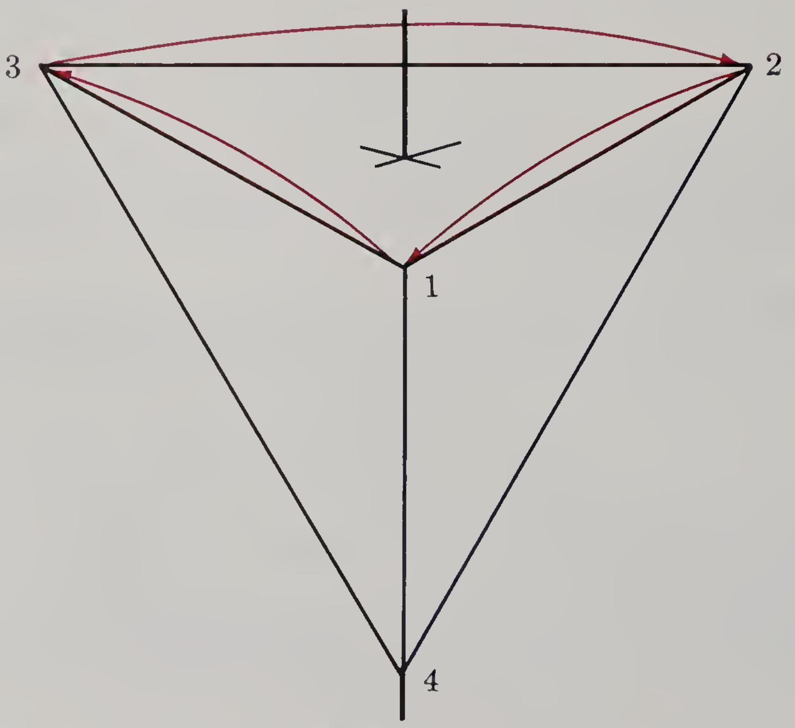
Thus, altogether there are $1 + 9 + 8 + 6$ or 24 rigid motions that map the cube into itself. It can be shown that these 24 motions form a group.

Rigid Motions of the Regular Tetrahedron

The two types of rigid motions of the regular tetrahedron (a regular tetrahedron has four faces that are equilateral triangles) are exhibited in Figure 8.4. The two types are the following.



(*a*) *Rotation about the line joining the centres of opposite edges*
3 such lines, 1 rotation about each—3 rotations



(*b*) *Rotation about the line joining the vertex to the centre of the opposite face*
4 such lines, 2 rotations about each—8 rotations

Figure 8.4

Type 1: rotation about a line joining the midpoints of opposite edges,

Type 2: rotation about a line joining a vertex to the centre of the opposite face.

Type 1: Figure 8.4 (*a*) indicates a rotation of 180° about the line joining the midpoints of opposite edges 1 3 and 2 4. This rotation produces the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

on the vertices of the tetrahedron.

There are three such lines and so there are three rigid motions of Type 1 of the regular tetrahedron (apart from the identity).

Type 2: Figure 8.4 (*b*) indicates a rotation of 120° about the line joining vertex 4 to the centre of face 123. Rotations of 120° and 240° are possible. Rotation of 360° or 0° is the identity. The rotation of 120° produces the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

on the vertices of the tetrahedron. (State the permutation produced by a rotation of 240°.)

There are four such lines and so there are 4×2 or eight rigid motions of Type 2 of the regular tetrahedron (apart from the identity).

Thus, altogether there are $1 + 3 + 8$ or 12 rigid motions that map the regular tetrahedron into itself. Again it can be shown that these 12 motions form a group.

Such groups as the above, and other similar geometric groups, have importance in crystallography and nuclear fundamental particle theory.

EXERCISE 8.16

1. Write the permutations corresponding to rotations of the cube about the line joining
   (a) centres of faces 1584 and 2673,
   (b) centres of faces 1234 and 5678,
   (c) opposite vertices 1 and 7,
   (d) opposite vertices 2 and 8,
   (e) opposite vertices 3 and 5,
   (f) midpoints of edges 15 and 37,
   (g) midpoints of edges 12 and 87,
   (h) midpoints of edges 56 and 43,
   (i) midpoints of edges 14 and 67,
   (j) midpoints of edges 58 and 23.

2. Write the permutations corresponding to rotations of the regular tetrahedron about the line joining
   (a) midpoints of edges 12 and 34,
   (b) midpoints of edges 23 and 14,
   (c) vertex 1 to the centre of face 234,
   (d) vertex 2 to the centre of face 134,
   (e) vertex 3 to the centre of face 124.
3. Find the rotations which are the inverses of those in questions (1)(a), (c), (2)(a), (c).
4. Show that the rotations of the regular tetrahedron form a group.
5. Show that the following sets are groups.
   (a) The Type 1 rigid motions of the cube
   (b) The Type 2 rigid motions of the cube
   (c) The Type 3 rigid motions of the cube
   (d) The Type 1 rigid motions of the regular tetrahedron
   (e) The Type 2 rigid motions of the regular tetrahedron

Chapter Summary

Permutations · Motions of a plane figure · Translations, rotations, reflections · Functions · 2 × 2 real matrices · Binary operations · Multiplication of the elements of a set · Products of permutations, motions, transformations, etc. · Multiplication tables · Groupoids · Semigroups · Identity or neutral elements · Inverses · Groups · Rings · Fields · Rigid motions of the cube and the tetrahedron

REVIEW EXERCISE 8

1. Determine which of the group postulates are satisfied by the following algebraic systems.
   (a) Ra with $a \Theta b = |ab|$ (b) I with $a \Theta b = aba$
   (c) $\left\{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\right\}$
   under the multiplication of permutations
   (d) $\left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}\right\}$
   under the multiplication of permutations
   (e) The set of all integral multiples of 4 under ordinary multiplication
   (f) The set $\{0, 1, 2, 3\}$ with Θ as defined by the following table

Θ	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(g) The set $\{0, 1, 2, 3\}$ with Θ as defined by the following table

Θ	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	0
3	0	3	0	1

2. Determine which of the ring postulates are satisfied by the following algebraic systems.
   (a) All integral multiples of 4 under ordinary addition and multiplication
   (b) All rationals of the form $a/5$ under ordinary addition and multiplication
   (c) All complex numbers $a + bi$ with $a^2 + b^2 = 1$ under ordinary addition and multiplication
   (d) All complex numbers $a + ai$ under ordinary addition and multiplication
   (e) I with ordinary addition but $a \otimes b = 0$ for all a, b
   (f) The set $\{0, 1, 2, 3\}$ with addition defined as in question (1)(f) and multiplication defined as in question (1)(g).

3. Which of the systems in question (2) have
   (a) commutative multiplication?
   (b) unit elements?

4. Which of the systems in question (2) are fields? If a system is not a field, state which field postulates do not hold.

TRIGONOMETRIC FUNCTIONS

Angle		Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
Degrees	Radians						
0°	.0000	.0000	1.0000	.0000	undefined	1.000	undefined
1°	.0175	.0175	.9998	.0175	57.2900	1.000	57.30
2°	.0349	.0349	.9994	.0349	28.6363	1.001	28.65
3°	.0524	.0523	.9986	.0524	19.0811	1.001	19.11
4°	.0698	.0698	.9976	.0699	14.3007	1.002	14.34
5°	.0873	.0872	.9962	.0875	11.4301	1.004	11.47
6°	.1047	.1045	.9945	.1051	9.5144	1.006	9.567
7°	.1222	.1219	.9925	.1228	8.1443	1.008	8.206
8°	.1396	.1392	.9903	.1405	7.1154	1.010	7.185
9°	.1571	.1564	.9877	.1584	6.3138	1.012	6.392
10°	.1745	.1736	.9848	.1763	5.6713	1.015	5.759
11°	.1920	.1908	.9816	.1944	5 1446	1.019	5.241
12°	.2094	.2079	.9781	.2126	4.7046	1.022	4.810
13°	.2269	.2250	.9744	.2309	4.3315	1.026	4.445
14°	.2443	.2419	.9703	.2493	4.0108	1.031	4.134
15°	.2618	.2588	.9659	.2679	3.7321	1.035	3.864
16°	.2793	.2756	.9613	.2867	3.4874	1.040	3.628
17°	.2967	.2924	.9563	.3057	3.2709	1.046	3.420
18°	.3142	.3090	.9511	.3249	3.0777	1.051	3.236
19°	.3316	.3256	.9455	.3443	2.9042	1.058	3.072
20°	.3491	.3420	.9397	.3640	2.7475	1.064	2.924
21°	.3665	.3584	.9336	.3839	2.6051	1.071	2.790
22°	.3840	.3746	.9272	.4040	2.4751	1.079	2.669
23°	.4014	.3907	.9205	.4245	2.3559	1.086	2.559
24°	.4189	.4067	.9135	.4452	2.2460	1.095	2.459
25°	.4363	.4226	.9063	.4663	2.1445	1.103	2.366
26°	.4538	.4384	.8988	.4877	2.0503	1.113	2.281
27°	.4712	.4540	.8910	.5095	1.9626	1.122	2.203
28°	.4887	.4695	.8829	.5317	1.8807	1.133	2.130
29°	.5061	.4848	.8746	.5543	1.8040	1.143	2.063
30°	.5236	.5000	.8660	.5774	1.7321	1.155	2.000
31°	.5411	.5150	.8572	.6009	1.6643	1.167	1.942
32°	.5585	.5299	.8480	.6249	1.6003	1.179	1.887
33°	.5760	.5446	.8387	.6494	1.5399	1.192	1.836
34°	.5934	.5592	.8290	.6745	1.4826	1.206	1.788
35°	.6109	.5736	.8192	.7002	1.4281	1.221	1.743
36°	.6283	.5878	.8090	.7265	1.3764	1.236	1.701
37°	.6458	.6018	.7986	.7536	1.3270	1.252	1.662
38°	.6632	.6157	.7880	.7813	1.2799	1.269	1.624
39°	.6807	.6293	.7771	.8098	1.2349	1.287	1.589
40°	.6981	.6428	.7660	.8391	1.1918	1.305	1.556
41°	.7156	.6561	.7547	.8693	1.1504	1.325	1.524
42°	.7330	.6691	.7431	.9004	1.1106	1.346	1.494
43°	.7505	.6820	.7314	.9325	1.0724	1.367	1.466
44°	.7679	.6947	.7193	.9657	1.0355	1.390	1.440
45°	.7854	.7071	.7071	1.0000	1.0000	1.414	1.414

TRIGONOMETRIC FUNCTIONS

ANGLE		SINE	COSINE	TANGENT	COTANGENT	SECANT	COSECANT
Degrees	Radians						
45°	.7854	.7071	.7071	1.0000	1.0000	1.414	1.414
46°	.8029	.7193	.6947	1.0355	.9657	1.440	1.390
47°	.8203	.7314	.6820	1.0724	.9325	1.466	1.367
48°	.8378	.7431	.6691	1.1106	.9004	1.494	1.346
49°	.8552	.7547	.6561	1.1504	.8693	1.524	1.325
50°	.8727	.7660	.6428	1.1918	.8391	1.556	1.305
51°	.8901	.7771	.6293	1.2349	.8098	1.589	1.287
52°	.9076	.7880	.6157	1.2799	.7813	1.624	1.269
53°	.9250	.7986	.6018	1.3270	.7536	1.662	1.252
54°	.9425	.8090	.5878	1.3764	.7265	1.701	1.236
55°	.9599	.8192	.5736	1.4281	.7002	1.743	1.221
56°	.9774	.8290	.5592	1.4826	.6745	1.788	1.206
57°	.9948	.8387	.5446	1.5399	.6494	1.836	1.192
58°	1.0123	.8480	.5299	1.6003	.6249	1.887	1.179
59°	1.0297	.8572	.5150	1.6643	.6009	1.942	1.167
60°	1.0472	.8660	.5000	1.7321	.5774	2.000	1.155
61°	1.0647	.8746	.4848	1.8040	.5543	2.063	1.143
62°	1.0821	.8829	.4695	1.8807	.5317	2.130	1.133
63°	1.0996	.8910	.4540	1.9626	.5095	2.203	1.122
64°	1.1170	.8988	.4384	2.0503	.4877	2.281	1.113
65°	1.1345	.9063	.4226	2.1445	.4663	2.366	1.103
66°	1.1519	.9135	.4067	2.2460	.4452	2.459	1.095
67°	1.1694	.9205	.3907	2.3559	.4245	2.559	1.086
68°	1.1868	.9272	.3746	2.4751	.4040	2.669	1.079
69°	1.2043	.9336	.3584	2.6051	.3839	2.790	1.071
70°	1.2217	.9397	.3420	2.7475	.3640	2.924	1.064
71°	1.2392	.9455	.3256	2.9042	.3443	3.072	1.058
72°	1.2566	.9511	.3090	3.0777	.3249	3.236	1.051
73°	1.2741	.9563	.2924	3.2709	.3057	3.420	1.046
74°	1.2915	.9613	.2756	3.4874	.2867	3.628	1.040
75°	1.3090	.9659	.2588	3.7321	.2679	3.864	1.035
76°	1.3265	.9703	.2419	4.0108	.2493	4.134	1.031
77°	1.3439	.9744	.2250	4.3315	.2309	4.445	1.026
78°	1.3614	.9781	.2079	4.7046	.2126	4.810	1.022
79°	1.3788	.9816	.1908	5.1446	.1944	5.241	1.019
80°	1.3963	.9848	.1736	5.6713	.1763	5.759	1.015
81°	1.4137	.9877	.1564	6.3138	.1584	6.392	1.012
82°	1.4312	.9903	.1392	7.1154	.1405	7.185	1.010
83°	1.4486	.9925	.1219	8.1443	.1228	8.206	1.008
84°	1.4661	.9945	.1045	9.5144	.1051	9.567	1.006
85°	1.4835	.9962	.0872	11.4301	.0875	11.47	1.004
86°	1.5010	.9976	.0698	14.3007	.0699	14.34	1.002
87°	1.5184	.9986	.0523	19.0811	.0524	19.11	1.001
88°	1.5359	.9994	.0349	28.6363	.0349	28.65	1.001
89°	1.5533	.9998	.0175	57.2900	.0175	57.30	1.000
90°	1.5708	1.0000	.0000	undefined	.0000	undefined	1.000

THE LOGARITHMIC FUNCTION TO BASE 10

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

THE LOGARITHMIC FUNCTION TO BASE 10

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

POWERS, ROOTS, AND RECIPROCALS 1—100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000	51	2,601	132,651	7.141	3.708	.0196
2	4	8	1.414	1.260	.5000	52	2,704	140,608	7.211	3.733	.0192
3	9	27	1.732	1.442	.3333	53	2,809	148,877	7.280	3.756	.0189
4	16	64	2.000	1.587	.2500	54	2,916	157,464	7.348	3.780	.0185
5	25	125	2.236	1.710	.2000	55	3,025	166,375	7.416	3.803	.0182
6	36	216	2.449	1.817	.1667	56	3,136	175,616	7.483	3.826	.0179
7	49	343	2.646	1.913	.1429	57	3,249	185,193	7.550	3.849	.0175
8	64	512	2.828	2.000	.1250	58	3,364	195,112	7.616	3.871	.0172
9	81	729	3.000	2.080	.1111	59	3,481	205,379	7.681	3.893	.0169
10	100	1,000	3.162	2.154	.1000	60	3,600	216,000	7.746	3.915	.0167
11	121	1,331	3.317	2.224	.0909	61	3,721	226,981	7.810	3.936	.0164
12	144	1,728	3.464	2.289	.0833	62	3,844	238,328	7.874	3.958	.0161
13	169	2,197	3.606	2.351	.0769	63	3,969	250,047	7.937	3.979	.0159
14	196	2,744	3.742	2.410	.0714	64	4,096	262,144	8.000	4.000	.0156
15	225	3,375	3.873	2.466	.0667	65	4,225	274,625	8.062	4.021	.0154
16	256	4,096	4.000	2.520	.0625	66	4,356	287,496	8.124	4.041	.0152
17	289	4,913	4.123	2.571	.0588	67	4,489	300,763	8.185	4.062	.0149
18	324	5,832	4.243	2.621	.0556	68	4,624	314,432	8.246	4.082	.0147
19	361	6,859	4.359	2.668	.0526	69	4,761	328,509	8.307	4.102	.0145
20	400	8,000	4.472	2.714	.0500	70	4,900	343,000	8.367	4.121	.0143
21	441	9,261	4.583	2.759	.0476	71	5,041	357,911	8.426	4.141	.0141
22	484	10,648	4.690	2.802	.0455	72	5,184	373,248	8.485	4.160	.0139
23	529	12,167	4.796	2.844	.0435	73	5,329	389,017	8.544	4.179	.0137
24	576	13,824	4.899	2.884	.0417	74	5,476	405,224	8.602	4.198	.0135
25	625	15,625	5.000	2.924	.0400	75	5,625	421,875	8.660	4.217	.0133
26	676	17,576	5.099	2.962	.0385	76	5,776	438,976	8.718	4.236	.0132
27	729	19,683	5.196	3.000	.0370	77	5,929	456,533	8.775	4.254	.0130
28	784	21,952	5.292	3.037	.0357	78	6,084	474,552	8.832	4.273	.0128
29	841	24,389	5.385	3.072	.0345	79	6,241	493,039	8.888	4.291	.0127
30	900	27,000	5.477	3.107	.0333	80	6,400	512,000	8.944	4.309	.0125
31	961	29,791	5.568	3.141	.0323	81	6,561	531,441	9.000	4.327	.0123
32	1,024	32,768	5.657	3.175	.0312	82	6,724	551,368	9.055	4.344	.0122
33	1,089	35,937	5.745	3.208	.0303	83	6,889	571,787	9.110	4.362	.0120
34	1,156	39,304	5.831	3.240	.0294	84	7,056	592,704	9.165	4.380	.0119
35	1,225	42,875	5.916	3.271	.0286	85	7,225	614,125	9.220	4.397	.0118
36	1,296	46,656	6.000	3.302	.0278	86	7,396	636,056	9.274	4.414	.0116
37	1,369	50,653	6.083	3.332	.0270	87	7,569	658,503	9.327	4.431	.0115
38	1,444	54,872	6.164	3.362	.0263	88	7,744	681,472	9.381	4.448	.0114
39	1,521	59,319	6.245	3.391	.0256	89	7,921	704,969	9.434	4.465	.0112
40	1,600	64,000	6.325	3,420	.0250	90	8,100	729,000	9.487	4.481	.0111
41	1,681	68,921	6.403	3.448	.0244	91	8,281	753,571	9.539	4.498	.0110
42	1,764	74,088	6.481	3.476	.0238	92	8,464	778,688	9.592	4.514	.0109
43	1,849	79,507	6.557	3.503	.0233	93	8,649	804,357	9.644	4.531	.0108
44	1,936	85,184	6.633	3.530	.0227	94	8,836	830,584	9.695	4.547	.0106
45	2,025	91,125	6.708	3.557	.0222	95	9,025	857,375	9.747	4.563	.0105
46	2,116	97,336	6.782	3.583	.0217	96	9,216	884,736	9.798	4.579	.0104
47	2,209	103,823	6.856	3.609	.0213	97	9,409	912,673	9.849	4.595	.0103
48	2,304	110,592	6.928	3.634	.0208	98	9,604	941,192	9.899	4.610	.0102
49	2,401	117,649	7.000	3.659	.0204	99	9,801	970,299	9.950	4.626	.0101
50	2,500	125,000	7.071	3.684	.0200	100	10,000	1,000,000	10.000	4.642	.0100

ANSWERS

Chapter 1

EXERCISE 1.1 (page 3)

1. (a), (b), (d), (e) are sets; (c), (f) are not.

2. (a) 16. (b) 12. (c) False, false, true.

3. (a) True, false, true. (b) False, true.

EXERCISE 1.2 (page 4)

1. (a) $\{a, b, c, d, e, f, g, h\}$. (b) $\{c, d\}$. (c) $\{e\}$.
(d) $\{c, d, e\}$. (e) $\{c, d, e\}$.

2. (a) $\{\frac{1}{5}\}$. (b) $\{\frac{1}{5}\}$. (c) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$.
(d) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$. (e) $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}\}$.
(f) $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}$. (g) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$. (h) $\{\frac{1}{3}, \frac{1}{7}, \frac{1}{9}\}$.
(i) $\{\frac{1}{3}, \frac{1}{7}, \frac{1}{9}\}$. (j) $\{\frac{1}{4}, \frac{1}{6}, \frac{1}{10}\}$. (k) $\{\frac{1}{4}, \frac{1}{6}, \frac{1}{10}\}$.

3. (a) $\{-4, -3, -2, 3, 4\}$. (b) ϕ. (c) $U = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.
(d) $\{-1, 0, 1, 2\} = P$. (e) U.

6. P' = set consisting of all composite positive integers and the integer 1.
Q' = set of all odd positive integers.
$P \cap Q = \{2\}$.
$P \cup Q$ = set consisting of all positive prime integers and all positive even integers.

7. $\{(-1, 2)\}$. 8. ϕ.

EXERCISE 1.5 (page 16)

8. Dual identities: (1) $(S \cup \phi)' \cap S' = S'$. (2) $(U \cup A) \cap B = B$.
(3) $A' \cap (A \cup B) = A' \cap B$. (4) $X \cup Y \cup X' = U$.
(5) $(A \cup B')' \cap B = A' \cap B$. (6) $(P \cap Q') \cup (P' \cup Q)' = P \cap Q'$.
(7) $(M' \cup U)' \cup M = M$.

9. U. **10.** ϕ. **11.** P. **12.** $A' \cup B$. **13.** N. **14.** U. **15.** ϕ. **16.** B.

EXERCISE 1.7 (page 21)

1. $A \times B = \{(a, r), (a, s), (b, r), (b, s), (c, r), (c, s), (d, r), (d, s)\}$.
$B \times A = \{(r, a), (r, b), (r, c), (r, d), (s, a), (s, b), (s, c), (s, d)\}$.

2. $P \times P = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$.

3. $\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)\}$.

4. $25 > 20$.

5. (a) $\{(-2,0), (-2,1), (0,0), (0,1), (-1,0), (-1,1), (1,0), (1,1), (2,0), (2,1)\}$.
(b) $\{(1,0), (2,0), (3,0)\}$.
(c) $\{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$.
(d) $\{(2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (5,1), (5,2), (5,3), (7,1), (7,2), (7,3), (11,1), (11,2), (11,3)\}$.

6. (a) 5, 2, 10. (b) 3, 1, 3. (c) 1, 9, 9. (d) 5, 3, 15. **7.** None. **8.** None.

9. $A \times B \times C = \{(-1,1,0), (-1,1,1), (-1,2,0), (-1,2,1), (0,1,0), (0,1,1), (0,2,0), (0,2,1), (1,1,0), (1,1,1), (1,2,0), (1,2,1)\}$.

EXERCISE 1.8 (page 22)

1.

	$n(A \times B)$	$n(A \cap B)$	$n(A \cup B)$
(a)	25	3	7
(b)	12	0	7
(c)	15	3	5
(d)	30	1	10
(e)	0	0	9

2. $n(P \cup Q) = n(P) + n(Q)$,
$n(P \cap Q) = 0$.

3. $n(T \cap S) = n(T)$,
$n(T \cup S) = n(S)$.

4. $B \subseteq A$. **5.** 0.

6. 5.

EXERCISE 1.9 (page 25)

1. 80 and 7. **2.** (a) 40 and 6. (b) 60 and 5. (c) 18 and 8. (d) 12 and 5. (e) 0 and 3. **3.** 64 and 4. **4.** 5. **5.** 24. **6.** (i) 216. (ii) 120.
7. $n(P) + n(Q) + n(R)$. **8.** 512 and 8. **9.** 5.

EXERCISE 1.10 (page 27)

1. 0.

2. From the data, the number of families using either A or B or C is 10,239 and this exceeds the total number of families.

3. (a) 75. (b) 80. (c) 28. **4.** 84. **5.** (a) 100. (b) 150. **6.** 13.

REVIEW EXERCISE 1 (page 29)

1. $A \cap B = \{3, 5, 7, 11, 13\}$. $A \cup B = \{1, 2, 3, 5, 7, 11, 13\}$.
$C = \{3, 5, 7, 11, 13\}$.

2. (a) $\{3, 5, 7, 11, 13, 17, 19\}$. (b) $\{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.
(c) $\{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$.
(d) $\{2, 4, 6, 8, 10, 12, 14, 16, 18\}$.
(e) $\{1, 2, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$.
(f) $\{4, 6, 8, 10, 12, 14, 16, 18\}$. (g) $\{2\}$.

4. (a) U. (b) D. (c) $P' \cap Q$. (d) C'. (e) $R \cup S$.

5. (a) (i), (iii), (v) are true.

6. (a) 12. (b) 72. (c) 4. (d) 7. (e) 2. (f) 7.

7. 12 **9.** 20. **10.** 231.

Chapter 2

EXERCISE 2.1 (page 33)

1. (i) 12. (ii) 240. (iii) 840. (iv) 2730. (v) 120. (vi) 360.
2. 336. **3.** 42. **4.** 120. **5.** 720. **6.** 60. **7.** 48. **8.** 100.
9. 15. **10.** 8. **11.** (a) 40,320. (b) 10,080. (c) 30,240. **12.** 40,320.
13. 120. **14.** 504. **15.** 12. **16.** (a) 720. (b) 480. (c) 48.
17. 970,200. **18.** 90. **19.** 210.

EXERCISE 2.2 (page 37)

1. (a) 56. (b) 20. (c) 20. (d) 56. (e) 56. (f) 1140.
2. (a) $(n+1)!$ (b) $(n+6)!$ (c) $(n-r+1)!$
3. (a) $(n+4)(n+5)$. (b) n. (c) $n+1$. (d) $n-r+1$.
(e) $n-r$. (*f*) $(n-r+1)(n-r)$.
4. (a) 2. (b) 3. (c) 6. (d) 4. (e) 3. (f) 8.

EXERCISE 2.3 (page 40)

1. (a) 60. (b) 56. (c) 2520. (d) 3960.
2. (a) 180. (b) 420. (c) 360. (d) 2520. (e) 34,650. (f) 840.
3. 30. **4.** 120. **5.** (a) 3360. (b) 360. **6.** 300. **7.** 12.
8. 84. **9.** (a) 180. (b) 60. (c) 120. (d) 12. (e) 48.
10. (a) 48. (b) 24. **11.** 35. **12.** 120.

EXERCISE 2.4 (page 45)

1. (a) 28. (b) 10. (c) 91. (d) 1. (e) 1. (f) 1330.
(g) 1326. (h) 100. (i) 600. (j) 5005.
2. 6. **3.** 21. **4.** 16. **5.** (a) 220. (b) 1320. **6.** 91.
7. 54. **8.** $\frac{n}{2}(n-3)$. **9.** 45. **10.** 840. **11.** 1316. **12.** 6167.
13. 153. **14.** 1287. **15.** 55. **16.** 91. **17.** 27,720. **18.** $\{2\}$.
19. $\{8\}$. **20.** $\{20\}$. **23.** $\frac{52!}{13!39!}$. **24.** $\frac{52!}{13!13!13!13!}$. **25.** 56.
26. 35.

EXERCISE 2.5 (page 49)

1. 8, 32, 512, 16, 1024.
2. $\{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$.
3. 7. **4.** 31. **5.** 15.
6. (a) 4. (b) 128. (c) 64. (d) 1024. (e) 1024. (f) 128.
7. 18. **8.** 94. **9.** 144. **10.** 36. **11.** 30.

EXERCISE 2.6 (page 50)

1. 864,000 **2.** 12,700,800. **3.** 1,814,400. **4.** 28,800. **5.** 60,000.
6. 11. **7.** 43. **8.** 85. **9.** 918. **10.** 19.

REVIEW EXERCISE 2 (page 52)

1. (a) 210. (b) 2520. (c) 9900. (d) 1320. (e) 35. (f) 1.
(g) 100. (h) 792. (i) 132. (j) 140. (k) 840. (l) 6510.
2. (a) x. (b) $x(x+1)$. (c) $n-r$.
6. (a) 462 (b) 200. (c) 461. **7.** (a) 1,680. (b) 100,800.
8. (a) 10,080. (b) 720. **9.** 210. **10.** 140.
11. 495,000 (if letter is in first position). **12.** 240. **13.** 55.
14. 840. **15.** 144. **16.** 15. **17.** 30. **18.** 34. **19.** 495.
20. 5,940. **21.** 27,720. **22.** 5775. **23.** 1440. **24.** 20. **25.** 10.
26. (a) $\{10\}$. (b) $\{3\}$. (c) $\{5\}$. (d) $\{12\}$.
27. 62. **28.** 199. **29.** 52. **30.** 23.

Chapter 3

EXERCISE 3.1 (page 56)

1. (a) $1+2+3+4+5+6$. (b) $1+2+3+4+5+6$.
(c) $1+0+(-1)+(-2)+(-3)+(-4)+(-5)$.
(d) $5+7+9+11+13+15+17+19$. (e) $3+6+9+12$.
(f) $|1|+|0|+|-1|+|-2|+|-3|+|-4|+|-5|$.
(g) $(-1)+(-1)^2+(-1)^3+(-1)^4+(-1)^5+(-1)^6$.
(h) $1^2+2^2+3^2+\cdots+(n-1)^2+n^2$.
(i) $\dfrac{1\cdot 2}{2}+\dfrac{2\cdot 3}{2}+\dfrac{3\cdot 4}{2}+\cdots+\dfrac{(n-1)n}{2}+\dfrac{n(n+1)}{2}$.
3. (a) 36. (b) 55. (c) 31. (d) 64. (e) 0. (f) -1.
4. (a) $\sum_{n=1}^{6} n$. (b) $\sum_{n=1}^{7} n^2$. $\sum_{n=1}^{6} 2^n$.
(d) $\sum_{n=1}^{6} (-1)^{n-1}2^n$. (e) $\sum_{i=1}^{n} a_i^2$. (f) $\sum_{r=1}^{4} a^r b^r$.
(g) $\sum_{n=1}^{5} a^n b^{n-1}$. (h) $\sum_{n=1}^{6} (4n-1)$. (i) $\sum_{n=1}^{5} 2(3^{n-1})$.

EXERCISE 3.2 (page 60)

1. (a) $3\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$. (b) $2\sum_{k=1}^{10} k^2 - 6\sum_{k=1}^{10} k$. (c) $\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 - 5n$.
(d) $\sum_{p=1}^{n} 2^p + 3\sum_{p=1}^{n} p$. (e) $\sum_{i=1}^{10} (-1)^i i^2 + 2\sum_{i=1}^{10} (-1)^i i$.
(f) $2\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k - 2n$.

2. (a) $6\sum_{k=1}^{n}k - 2n$. (b) 0.

3. (a) $\sum_{k=1}^{9}(k+11)$. (b) $\sum_{a=1}^{11}\frac{a-6}{a}$. (c) $\sum_{n=1}^{6}\frac{n+3}{n}$.

(d) $\sum_{p=1}^{10}p$. (e) $\sum_{k=1}^{4}2(k+6)^2$. (f) $\sum_{i=1}^{6}(i^2-5i+6)$.

EXERCISE 3.3 (page 63)

1. $\frac{n}{2}(n+1)$. **2.** $\frac{n}{n+1}$. **3.** $\frac{n}{3n+1}$. **4.** $\frac{2n}{2n+1}$.

5. $\frac{n}{6n+4}$. **6.** $\frac{n}{2}(3n+5)$. **7.** $\frac{n}{4(2n+1)}$. **8.** $3n-2$.

9. 2^{n-1}. **10.** $\frac{n}{n+1}$. **11.** $2^{n+1}-1$.

12. $\frac{(n+1)^2-1}{n+1} = 1 - \frac{1}{n+1}$. **13.** $\frac{2(3^n)+1}{3^n} = 2 + \frac{1}{3^n}$.

14. $\frac{n}{2}(n+1)$. **15.** $3^n - 2^n$.

16. (a) $t_1 = 1,\ t_2 = 4,\ t_3 = 7,\ t_4 = 16$, etc. (b) For example, $t_n = 3n-2$.

EXERCISE 3.5 (page 74)

1. $\frac{n}{2}(3n+1)$. **2.** $2n(24-n)$. **3.** $n^2(n+1)$.

4. $\frac{n}{6}(2n^2+9n+1)$. **5.** $\frac{k}{2}(k+1)(k^2+k-1)$. **6.** $\frac{n}{3}(n^2-3n+8)$.

7. $\frac{n}{6}(n+1)(4n+5)$. **8.** $n(-2n^2+13n+5)$.

9. $\frac{n}{4}(n+1)(n+2)(n+3)$. **10.** $\frac{n}{12}(n+1)(n+2)(9n-1)$.

11. 1240. **12.** 270. **13.** 27,900. **14.** 3741. **15.** 1444.

16. 20,556. **17.** 928. **18.** 10,184 **19.** 317,660. **20.** 48,020.

21. 3,491,040. **22.** 1,546,230. **23.** $\frac{n^2(n+1)^2}{4}$.

24. (a) $\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. (b) $\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

EXERCISE 3.6 (page 77)

1. $\frac{1}{x-1}+\frac{1}{x+2}$. **2.** $\frac{1}{x-1}-\frac{1}{x+2}$. **3.** $\frac{2}{x-3}+\frac{1}{x+2}$.

4. $\frac{-2}{x-2}+\frac{3}{x-1}$. **5.** $\frac{-15}{3-2x}+\frac{7}{1-x}$. **6.** $\frac{3}{2k-1}-\frac{2}{2k+1}$.

7. $\frac{\frac{1}{3}}{3k-1} - \frac{\frac{1}{3}}{3k+2}$. **8.** $\frac{1}{5k-2} - \frac{1}{5k+3}$. **9.** $\frac{n}{n+1}$.

10. $\frac{n}{2n+1}$. **11.** $\frac{4n}{3(4n+3)}$. **12.** $\frac{n}{3(4n+3)}$. **13.** $\frac{n}{3(5n+3)}$.

14. $\frac{3n}{2(3n+2)}$. **15.** $\frac{4n}{1-4n^2}$. **16.** $\frac{-3(3n+5)}{4(3n-1)(3n+2)}$.

REVIEW EXERCISE 3 (page 78)

1. (a) $(-1)+1+3+5+7+9$. (b) $8-10+12-14+16-18+20$.
(c) $3+2+1+0+(-1)+(-2)+(-3)+(-4)$.
(d) $10+5+2+1+2+5+10$.

2. (a) $\sum_{p=1}^{8}(-1)^{p+1}p$. (b) $\sum_{p=1}^{9}(4p-1)$. (c) $\sum_{p=1}^{7}3^{p-3}$.
(d) $\sum_{p=1}^{5}(a+b)^{p-3}$. (e) $\sum_{p=1}^{5}\frac{1}{(p+2)(p+3)}$. (f) $\sum_{p=1}^{n}\frac{1}{p(2p+1)}$.

3. (a) $\sum_{p=1}^{6}(2p+3)$. (b) $\sum_{k=1}^{6}(k-3)^2$. (c) $\sum_{k=1}^{n+1}2^{k-1}$.
(d) $\sum_{p=1}^{n}p(p+2)$. (e) $\sum_{k=1}^{5}k$. (f) $\sum_{i=1}^{n}(-1)^i 3(i-2)$.

5. (a) $\frac{n}{2}(n+3)$. (b) $\frac{n}{2}(3n+1)$. (c) $\frac{n}{2}(2n^2+n+1)$.
(d) $\frac{n}{6}(4n^2+9n-1)$. (e) $\frac{n}{4}(n-1)(n^2+3n-2)$. (f) $n(n^3-n-3)$.
(g) 366. (h) 2540. (i) 5875. (j) 1111. (k) 37,680. (l) 111,564.

9. (a) $\frac{2}{x-1}+\frac{3}{x+2}$. (b) $\frac{2}{x+3}-\frac{1}{x-2}$.
(c) $\frac{2}{1-x}+\frac{2}{1+x}$. (d) $\frac{4/3}{x-1}-\frac{4/3}{x+2}$.

10. (a) $\frac{n}{4(2n+1)}$. (b) $\frac{n}{3n+2}$. (c) $\frac{5k}{4(5k+4)}$.
(d) $\frac{(n-1)(3n-2)}{2n(n+1)}$. (e) $\frac{4n}{1-4n^2}$. (f) $\frac{12}{65}$.

Chapter 4

EXERCISE 4.1 (page 83)

1. (a) 2. (b) 5. (c) 3. (d) 5.

2. (a)
$$\sum_{r=0}^{4}\binom{4}{r}a^{4-r}b^r = \binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4$$
$$= a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$
(b)
$$\sum_{r=0}^{5}\binom{5}{r}a^{5-r}b^r = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

(c) $\sum_{r=0}^{8}\binom{8}{r}x^{8-r}y^r = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8.$

(d) $\sum_{r=0}^{7}\binom{7}{r}p^{7-r}q^r = p^7 + 7p^6q + 21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 21p^2q^5 + 7pq^6 + q^7.$

3. (a) 5. (b) 6. (c) 9. (d) 8.

4. (a) 12. (b) 16. (c) 19. (d) 21.

EXERCISE 4.2 (page 85)

1. (a) $\sum_{r=0}^{12}\binom{12}{r}x^{12-r}y^r.$ (b) $\sum_{r=0}^{9}(-1)^r\binom{9}{r}x^{9-r}y^r.$ (c) $\sum_{r=0}^{7}\binom{7}{r}(2x)^{7-r}y^r.$

(d) $\sum_{r=0}^{8}\binom{8}{r}(2x)^{8-r}\left(\frac{y}{2}\right)^r.$ (e) $\sum_{r=0}^{10}\binom{10}{r}x^{20-2r}.$ (f) $\sum_{r=0}^{15}(-2)^r\binom{15}{r}x^{2r}.$

2. (a) $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$

(b) $16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4.$

(c) $64x^6 - 96x^5y + 60x^4y^2 - 20x^3y^3 + \frac{15}{4}x^2y^4 - \frac{3}{8}xy^5 + \frac{1}{64}y^6.$

(d) $x^8 + 4x^5 + 6x^2 + \frac{4}{x} + \frac{1}{x^4}.$ (e) $256x^8 - 128x^7 + 24x^6 - 2x^5 + \frac{1}{16}x^4.$

(f) $243a^5 - 810\frac{a^4}{b} + 1080\frac{a^3}{b^2} - 720\frac{a^2}{b^3} + 240\frac{a}{b^4} - \frac{32}{b^5}.$

3. (a) $a^{10} + 10a^9b + 45a^8b^2 + \cdots.$ (b) $a^8 - 8a^7b + 28a^6b^2 + \cdots.$

(c) $a^7 + 21a^6b + 189a^5b^2 + \cdots.$ (d) $1 - 45x + 945x^2 + \cdots.$

(e) $243a^5 - 810a^4 + 1080a^3 \cdots.$ (f) $1 + 24b + 264b^2 + \cdots.$

(g) $256x^8 - 256x^7y + 112x^6y^2 + \cdots.$ (h) $x^{12} - 12x^9 + 60x^6 + \cdots.$

(i) $729x^{12} - 1458\frac{x^{11}}{y} + 1215\frac{x^{10}}{y^2} + \cdots.$

4. (a) $210a^4b^6.$ (b) $-280a^4b^3.$ (c) $\frac{15}{4}x^{10}.$ (d) $-960x^{11}.$

(e) $\frac{280}{81}x^3y^4.$ (f) $4320x^3y^3.$ (g) 70. (h) $\binom{2n}{n}.$

5. $84x^5y^2.$ **6.** $-448x^6y^5.$ **7.** $2160x^4y^{-8}.$ **8.** $a = 2, b = -\frac{1}{2}.$

9. $a = 1, b = -\frac{2}{3}, n = 8.$

EXERCISE 4.3 (page 90)

1. (a) $\binom{12}{r}x^r.$ (b) $(-1)^r2^{8-r}\binom{8}{r}a^r.$ (c) $\binom{7}{r}x^{14-r}.$

(d) $\binom{7}{r}a^{7-2r}.$ (e) $(-1)^r\binom{8}{r}p^{8-2r}.$ (f) $(-1)^r\binom{10}{r}x^{20-3r}.$

(g) $\binom{10}{r}2^{10-r}\,a^{10-3r}.$ (h) $(-1)^r\binom{15}{r}2^{15-r}a^{15-4r}.$ (i) $\binom{12}{r}2^r\,a^{3n-5r}.$

2. 12, 220, 792. **3.** $t_4 = 455.$ **4.** $t_2 = 5120x^8$; $t_6 = 8064$; $t_8 = 960x^{-4}.$

5. $t_3 = \frac{1215}{4}$; $t_5 = 540x^{-6}.$ **6.** $t_4 = -84a^3$; $t_6 = -126a^{-7}$; no term of form $ka^{-10}.$

7. $t_7 = \frac{21}{2}.$ 8. 8. **9.** 9. **10.** 10.

REVIEW EXERCISE 4 (page 91)

1. (a) $\sum_{r=0}^{7}\binom{7}{r}a^{7-r}b^{r}$. (b) $\sum_{r=0}^{n}\binom{n}{r}2^{r}x^{n-r}y^{r}$.

(c) $\sum_{r=0}^{10}(-1)^{r}\binom{10}{r}2^{10-r}a^{10-r}b^{r}$. (d) $\sum_{r=0}^{8}(-1)^{r}\binom{8}{r}x^{16-3r}$.

(e) $\sum_{r=0}^{15}\binom{15}{r}2^{15-2r}x^{30-3r}$. (f) $\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}2^{n-r}x^{2r}$.

2. (a) $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
(b) $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.
(c) $32a^5 + 240a^4b + 720a^3b^2 + 1080a^2b^3 + 810ab^4 + 243b^5$

3. (a) $a^{12} + 24a^{11}b + 264a^{10}b^2 + \cdots$.
(b) $256x^8 - 1024x^7y + 1792x^6y^2 + \cdots$.
(c) $1024x^{20} + 2560x^{19} + 2880x^{18} + \cdots$.
(d) $729x^6 - 1458x^4 + 1215x^2 + \cdots$.
(e) $\frac{1}{1024}x^{10} + \frac{5}{128}x^8 + \frac{45}{64}x^6 + \cdots$.
(f) $a^{20}x^{40} + 20a^{19}bx^{38} + 190a^{18}b^2x^{36} + \cdots$.

4. $594x^{-8}$. **5.** $210x^2$. **6.** $t_9 = 180x^2$. **7.** $32a^{13} + 176a^{12} + 336a^{11} + \cdots$.

8. $16 - 80x + 136x^2 + \cdots$. **9.** $t_3 = 32805x^{12}$; $t_6 = 252x^{15}$.

10. $t_3 = 240$. **11.** $t_6 = 252$. **12.** $p = -3, n = 12$.

13. $p = \frac{1}{2}, n = 10$. **14.** 8. **15.** $a = 2, b = -\frac{1}{4}$. **16.** $a = 3, b = \frac{1}{3}, n = 6$. **17.** nx^{n-1}.

18. (a) $1 + 1 + \frac{1}{2!}(1 - n) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \frac{1}{4!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)$.
(b) 2.7.

Chapter 5

EXERCISE 5.1 (page 95)

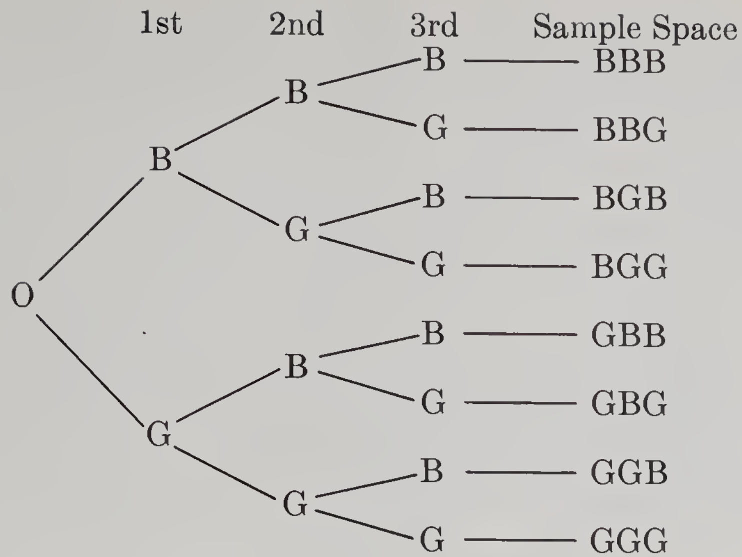
1. (a) $\frac{1}{6}$. (b) $\frac{1}{3}$. (c) $\frac{1}{2}$. (d) $\frac{1}{2}$. (e) $\frac{1}{2}$. (f) $\frac{1}{2}$.
2. (a) $\frac{5}{8}$. (b) $\frac{3}{8}$. **3.** (a) $\frac{2}{7}$. (b) $\frac{3}{7}$. (c) $\frac{4}{7}$. **4.** $\frac{5}{9}$.
5. (a) $\frac{19}{63}$. (b) $\frac{4}{21}$. (c) $\frac{32}{63}$. **6.** (a) $\frac{1}{15}$. (b) $\frac{1}{30}$. (c) $\frac{1}{5}$.
7. (a) $\frac{4}{9}$. (b) $\frac{5}{9}$. (c) $\frac{1}{9}$. **8.** $\frac{3}{8}$. **9.** (a) $\frac{1}{22}$. (b) $\frac{9}{22}$.
10. (a) $\frac{1}{36}$. (b) $\frac{1}{12}$. **11.** $\frac{1}{8}$. **12.** (a) $\frac{1}{17}$. (b) $\frac{1}{221}$. (c) $\frac{25}{102}$. (d) $\frac{1}{1326}$.

EXERCISE 5.2 (page 99)

1. $S_1 = \{(0, 3), (1, 2), (2, 1), (3, 0)\}$, number of heads and tails.
$S_2 = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.

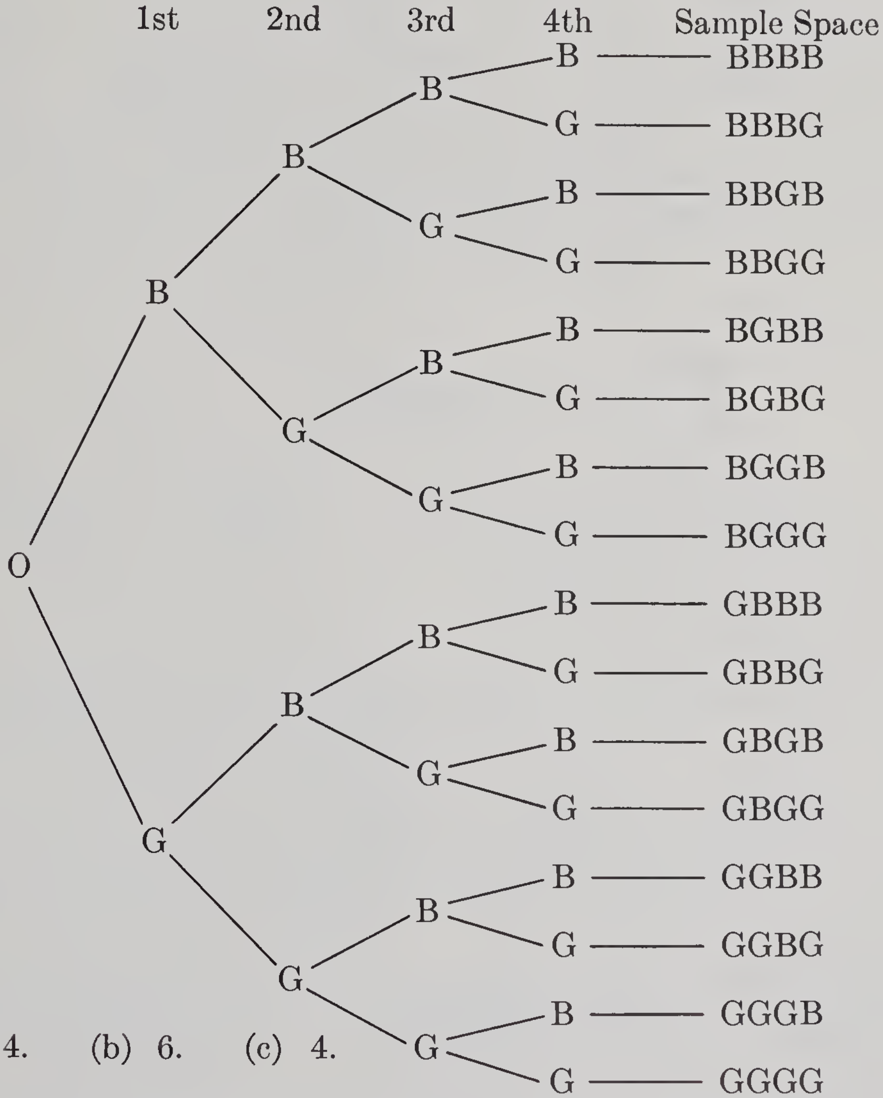
2. One in S_1, three in S_2.

3.



(a) 1. (b) 3.

4.



(a) 4. (b) 6. (c) 4.

5. $S = \{$RBG, RBW, RBY, RGW, RGY, RWY, BGW, BGY, BWY, GWY$\}$.
(a) 6. (b) 4. (c) 3. (d) 9.

6. $S = \{$BB, BG, BR, GB, GG, GR, RB, RG, RR$\}$. (a) 3. (b) 2. (c) 8.

7. {1234, 3124, 1234, 3142, 1324, 3214, 1342, 3241, 1423, 3412, 1432, 3421, 2134, 4123, 2143, 4132, 2314, 4213, 2341, 4231, 2413, 4312, 2431, 4321.}
(a) 6. (b) 7. (c) 9.

EXERCISE 5.3 (page 105)

1. (a) $\frac{2}{9}$. (b) $\frac{1}{6}$. (c) $\frac{5}{12}$. (d) $\frac{5}{12}$. (e) $\frac{5}{36}$. (f) $\frac{1}{6}$.
(g) $\frac{1}{3}$. (h) $\frac{1}{18}$. (i) $\frac{4}{9}$. (j) $\frac{5}{12}$. (k) $\frac{1}{3}$. (l) 0.
2. $\frac{74}{299}$. (b) $\frac{225}{299}$. (c) $\frac{37}{598}$. (d) $\frac{75}{299}$.
3. (a) $\frac{1}{8}$. (b) $\frac{3}{8}$. (c) $\frac{1}{2}$. (d) $\frac{7}{8}$.
4. (a) $\frac{1}{3}$. (b) $\frac{1}{15}$. (c) $\frac{8}{15}$. (d) $\frac{1}{5}$.
5. (a) $\frac{1}{4}$. (b) $\frac{3}{8}$. (c) $\frac{1}{4}$.
6. (a) $\frac{1}{4}$. (b) $\frac{7}{24}$. (c) $\frac{9}{24}$. (d) $\frac{1}{24}$.
7. (a) $\frac{3}{5}$. (b) $\frac{3}{10}$. (c) $\frac{2}{5}$. (d) $\frac{9}{10}$.

EXERCISE 5.4 (page 109)

1. (a) 7 to 4. (b) 4 to 7. **2.** (a) 1 to 5. (b) 1 to 2. (c) 5 to 1.
3. (a) 3 to 4. (b) 5 to 2. **4.** 4 to 3. 5. 14 to 25.
6. 1 to 3. **7.** 7 to 1. **8.** (a) 25 to 77. (b) 188 to 33. (c) 1 to 220.
9. (a) 1 to 8. (b) 31 to 5. (c) 5 to 7. (d) 5 to 1.
10. 6 to 1. **11.** 2 to 7. **12.** 1 to 1.

EXERCISE 5.5 (page 113)

1. (a) $\frac{1}{9}$. (b) $\frac{5}{9}$. (c) $\frac{3}{4}$. (d) $\frac{2}{9}$.
2. (a) $\frac{1}{4}$. (b) $\frac{1}{2}$. (c) $\frac{11}{20}$. (d) $\frac{4}{5}$.
3. (a) $\frac{5}{19}$. (b) $\frac{11}{19}$. **4.** $\frac{8}{15}$.
5. (a) $\frac{1}{2}$. (b) $\frac{4}{5}$. (c) $\frac{13}{20}$. (d) $\frac{2}{5}$. (e) $\frac{7}{20}$.
6. $\frac{11}{28}$. **7.** $\frac{55}{221}$. **8.** (a) $\frac{8}{15}$. (b) $\frac{1}{8}$. (c) $\frac{2}{15}$.
9. 22%; 121. **10.** $\frac{9}{20}$.

EXERCISE 5.6 (page 115)

1. (b), (e). **2.** (b). **3.** (d), (e). **4.** No. $P(A \cup B) = \frac{1}{2}$. **5.** Yes. $P(A \cup B) = \frac{2}{3}$.
6. (a) $\frac{11}{36}$. (b) $\frac{5}{18}$. (c) $\frac{1}{2}$. (d) $\frac{11}{36}$. (e) $\frac{2}{3}$. (f) $\frac{5}{9}$.
7. (a) $\frac{3}{4}$. (b) $\frac{3}{4}$. (c) $\frac{1}{2}$. (d) 1.
8. (a) $\frac{3}{8}$. (b) $\frac{3}{4}$. (c) $\frac{5}{8}$. (d) $\frac{5}{8}$. (e) $\frac{5}{8}$.
9. (a) $\frac{1}{17}$. (b) $\frac{25}{102}$. (c) $\frac{1}{221}$. (d) $\frac{1}{1326}$. (e) $\frac{55}{221}$. **10.** $\frac{79}{1326}$.

EXERCISE 5.7 (page 120)

1. (a) Independent. (b) Dependent. (c) Mutually exclusive.
(d) Dependent. (e) Dependent. (f) Dependent.
2. $\frac{1}{72}$. **3.** $\frac{1}{64}$. **4.** (a) $\frac{4}{9}$. (b) $\frac{1}{9}$. (c) $\frac{4}{9}$.
5. (a) Independent. (b) Mutually exclusive. (c) Dependent.
6. (a) $\frac{8}{225}$. (b) $\frac{8}{125}$. (c) $\frac{1}{27}$. (d) $\frac{16}{375}$.
7. (a) $\frac{1}{8}$. (b) $\frac{1}{8}$. (c) $\frac{5}{8}$. **8.** No. **9.** No. **10.** Yes.

EXERCISE 5.8 (page 125)

1. (a) $\frac{25}{216}$. (b) $\frac{5}{324}$. (c) $\frac{625}{1296}$. **2.** (a) $\frac{5}{16}$. (b) $\frac{1}{2}$. (c) $\frac{1}{32}$.
3. (a) $\frac{5}{16}$. (b) $\frac{1}{2}$. (c) $\frac{1}{32}$. (d) $\frac{31}{32}$. **4.** (a) $\frac{1}{16}$. (b) $\frac{135}{512}$.
5. $\frac{233}{729}$. **6.** (a) $\frac{1}{216}$. (b) $\frac{25}{72}$. (c) $\frac{91}{216}$. (d) $\frac{919}{5832}$.
7. (a) $\frac{27}{64}$. (b) $\frac{3}{64}$. (c) $\frac{243}{256}$. (d) $\frac{1}{256}$. **8.** $\frac{4547}{8192}$.
9. $\frac{1}{4}$. **10.** $\frac{1}{6}$. **11.** $\frac{27}{125}$. **12.** 0.189. **13.** 0.000547034.

REVIEW EXERCISE 5 (page 128)

1. (a) {ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF}.
(b) {A, A A, K A, Q A, J A, 10 K, A K, K K, Q K, J K, 10 Q, A Q, K Q, Q Q, J Q, 10 J, A J, K J, Q J, J J, 10 10, A 10, K 10, Q 10, J 10, 10}.
2. (a) $\frac{3}{5}$. (b) $\frac{3}{10}$. (c) $\frac{9}{10}$. (d) $\frac{1}{2}$. (e) $\frac{3}{5}$. (f) $\frac{1}{10}$.
3. (a) $\frac{5}{18}$. (b) $\frac{1}{36}$. (c) $\frac{5}{36}$. (d) $\frac{1}{12}$. (e) $\frac{1}{6}$. (f) $\frac{2}{3}$. (g) $\frac{1}{3}$.
4. (a) $\frac{1}{13}$. (b) $\frac{1}{26}$. (c) $\frac{1}{4}$. (d) $\frac{2}{13}$. (e) $\frac{5}{13}$.
5. (a) $\frac{1}{100}$. (b) $\frac{21}{23}$. (c) $\frac{6}{115}$. **6.** $\frac{43}{91}$.
7. (a) $\frac{7}{33}$. (b) $\frac{4}{165}$. (c) $\frac{28}{55}$. (d) $\frac{14}{55}$. **8.** 38,916 to 15,229.
9. 14 to 25.
10. (a) Independent. (b) Dependent. (c) Mutually exclusive. (d) Independent.
11. $\frac{3}{400}$. **12.** $\frac{2072}{3125}$. **13.** (a) $\frac{80}{243}$. (b) $\frac{20}{243}$. (c) $\frac{716}{729}$. (d) $\frac{656}{729}$. **14.** $\frac{241}{969}$.
15. (a) $\frac{15}{64}$. (b) $\frac{57}{64}$. (c) $\frac{7}{64}$. (d) $\frac{1}{64}$. **16.** 0.5786.

Chapter 6

EXERCISE 6.1 (page 133)

2. (a) $(4, 360°)$, $(-4, 180°)$. (b) $(0, 390°)$, $(0, 0°)$.
(c) $(5, 435°)$, $(-5, 255°)$. (d) $(-5, 450°)$, $(5, 270°)$.
(e) $(-3, -60°)$, $(3, 120°)$. (f) $\left(-10\sqrt{2}, \frac{3\pi}{4}\right)$, $\left(10\sqrt{2}, \frac{7\pi}{4}\right)$.
(g) $(-12\sqrt{3}, 240°)$, $(12\sqrt{3}, 60°)$. (h) $(-3, 30°)$, $(3, 210°)$.
(i) $(3, 210°)$, $(-3, 30°)$.
9. Two. The pole has an infinite number.
11. (a) Circle, centre the pole, radius 4. (b) Circle, centre the pole, radius a.
13. (a) Line through pole, making angle of 60° with polar axis.
(b) Line through pole, making angle θ_1 with polar axis.
14. Circle, centre $(1, 0°)$, radius 1. **15.** Circle, centre $(1, 90°)$, radius 1.
16. Circle, centre $(\frac{3}{2}, 90°)$, radius $\frac{3}{2}$. **17.** 163° 4′.
18. Straight line through $(2, 0°)$ perpendicular to polar axis.

EXERCISE 6.2 (page 137)

1. (a) $(4, 0)$. (b) $(0, -3)$. (c) $(1, 1)$. (d) $(0, 6)$.
(e) $(2\sqrt{3}, 2)$. (f) $(-1, \sqrt{3})$. (g) $(-\frac{5}{2}, \frac{5}{2}\sqrt{3})$. (h) $(2\sqrt{2}, -2\sqrt{2})$.
(i) $(3, 0)$. (j) $(0, 0)$. (k) $(-\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$. (l) $(0, 4)$.

2. (a) $(5, 0°)$. (b) $(2\sqrt{2}, 45°)$. (c) $(6, 90°)$. (d) $(4\sqrt{2}, 225°)$.
(e) $(3, 180°)$. (f) $(5, 270°)$. (g) $(0, 0°)$. (h) $(8, 300°)$.
(i) $(8, 150°)$. (j) $(5, 53°\,8')$. (k) $(13, 112°\,37')$. (l) $(5, 216°\,52')$.

3. (a) $r\cos\theta = 4$. (b) $r\sin\theta = -3$. (c) $r = 5$.
(d) $\tan\theta = \frac{3}{5}$. (e) $r(3\cos\theta - 4\sin\theta) + 6 = 0$. (f) $r^2\sin 2\theta = 8$.
(g) $r\sin^2\theta = 4\cos\theta$. (h) $r = 2\sin\theta$. (i) $r = 2\cos\theta + 6\sin\theta$.
(j) $r^2\cos 2\theta = 9$. (k) $r^2(5\cos^2\theta - 1) = 4$. (l) $r^2 \doteq \cos 2\theta$.

4. (a) $x^2 + y^2 = 36$. (b) $y = \frac{1}{\sqrt{3}}x$. (c) $x^2 + y^2 = 3x$.
(d) $y = 6$. (e) $x^2 + y^2 = 4x$. (f) $x^2 + y^2 = 2y$.
(g) $xy = 1$. (h) $x^2 - y^2 = 3$. (i) $x(x^2 + y^2) = 2y$.
(j) $x^2 + y^2 = 3y$. (k) $(x^2 + y^2)^{3/2} = 2xy$. (l) $y^2(x^2 + y^2) = a^2x^2$.

5. (a) Parabola. (b) Ellipse. (c) Hyperbola.

EXERCISE 6.3 (page 141)

2. (a) $\theta = 45°, 135°$. (b) $\theta = 30°, 90°, 150°$. (c) $\theta = 90°$.
(d) $\theta = 180°$. (e) $\theta = 60°$. (f) $\theta = 45°$.
(g) None. (h) $\theta = 0°$. (i) $\theta = 120°$.

3. (a) Symmetric with respect to polar axis, pole, and $\theta = \frac{\pi}{2}$.
(b) Symmetric w.r.t. polar axis. (c) Same as (a).
(d) Same as (a). (e) Symmetric w.r.t. pole.
(f) Same as (a). (g) Symmetric w.r.t. polar axis.
(h) Same as (a). (i) Symmetric w.r.t. polar axis.
(j) Symmetric w.r.t. polar axis. (k) Symmetric w.r.t. $\theta = \frac{\pi}{2}$.
(l) Symmetric w.r.t. pole. (m) Same as (a).
(n) No symmetry w.r.t. pole, polar axis, $\theta = 0°$, or $\theta = \frac{\pi}{2}$.
(o) Symmetric w.r.t. polar axis. (p) Symmetric w.r.t. $\theta = \frac{\pi}{2}$.
(q) Symmetric w.r.t. polar axis. (r) Symmetric w.r.t. $\theta = \frac{\pi}{2}$.

4. (a) $180° < \theta < 360°$. (b) $45° < \theta < 135°$, $225° < \theta < 315°$.
(c) $120° < \theta < 240°$. (d) $90° \leq \theta \leq 270°$.
(e) $90° \leq \theta < 180°$, $270° \leq \theta < 360°$. (f) None.
(g) $\theta = 270°$. (h) $90° \leq \theta \leq 270°$. (i) $180° < \theta < 360°$.

5. (a) Pole. (b) $(1, 0°)$, $(-1, 0°)$.
(c) $(-1 + \sqrt{3}, 0°)$, $(-1 - \sqrt{3}, 0°)$. (d) $(2, 0°)$, $(-2, 0°)$.
(e) Pole. (f) Pole. (g) $(\sqrt{2}, 0°)$, $(-\sqrt{2}, 0°)$.
(h) Pole. (i) Pole.

6. (a) $(\frac{3}{2}, 60°)$, $(\frac{3}{2}, 300°)$. (b) $(\sqrt{2}, 45°)$, $(0, 0°)$.
(c) $(2, 30°)$, $(2, 150°)$, $(0, 0°)$.

REVIEW EXERCISE 6 (page 150)

2. (a) About polar axis. (b) About pole. (c) About $\theta = \frac{\pi}{2}$.
(d) About pole.

3. (a) $(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$. (b) $(\sqrt{3}, 1)$. (c) $(-5, 0)$. (d) $(0, 3)$.

4. (a) $(6, 90°)$. (b) $(-2, 0°)$. (c) $(4, 120°)$. (d) $(8, 210°)$.
5. (a) $r = 2\cos\theta$. (b) $\tan\theta = \frac{2}{5}$. (c) $r\sin^2\theta + 9\cos\theta = 0$.
6. (a) $y = 2$. (b) $(x^2 + y^2)^2 = 2xy$. (c) $y^2(x^2 + y^2) = 4x^2$.
7. (a) Circle, radius, 2, centre $(2, 90°)$. (b) Circle, radius 4, centre at pole.
(c) Circle, radius 2, centre $(2, 180°)$. (d) Line $y = 4$.
(e) Line $x = -5$. (f) Line $x = 2$.
9. True if $f(2\theta_1 - \theta) = f(\theta)$.

Chapter 7

EXERCISE 7.1 (page 154)

1. (a) $6, -\sqrt{2}$. (b) $\frac{1}{2}, \frac{1}{2}$. (c) $-\pi, 6$. (d) $2x, 3y$.
(e) $4, 3$. (f) $0, -5$. (g) $\frac{1}{2}, -\sqrt{2}$. (h) $5\sqrt{2} + 3, 0$.
2. (a) $3 - 2i$. (b) $0 + 6i$. (c) $2 + (3 + \sqrt{2})i$. (d) $x + yi$.
3. (a) $(5, -\frac{3}{2})$. (b) $(-\sqrt{2}, 3)$. (c) $(\frac{2}{3}, -\pi)$. (d) (p, q).
5. (a) $2\sqrt{13}$. (b) $\sqrt{13}$. (c) $\frac{1}{2}\sqrt{10}$. (d) $2\sqrt{5}$.
(e) 3. (f) 3. (g) 2. (h) $\sqrt{\pi^2 + 16}$.
7. (a) 5. (b) $\sqrt{29}$. (c) 5. (d) $2\sqrt{10}$. (e) 4. (f) 3.
(g) $\sqrt{11}$. (h) 5.
8. $0 + 0i$. **9.** The imaginary axis. **10.** The real axis.
11. (a) Line $x = y$: (b) Line $x + y = 0$.

EXERCISE 7.2 (page 157)

1. (a) $x = \frac{2}{3}, y = \frac{2}{3}$. (b) $x = -\pi, y = \frac{1}{3}\sqrt{2}$. (c) $x = -\frac{1}{2}, y = \frac{11}{2}$.
(d) $x = 3, y = -3$. (e) $x = 0, y = -3$. (f) $x = -2, y = 1$.
(g) $x = \frac{7}{5}, y = \frac{8}{5}$. (h) $x = -\frac{10}{9}, y = -\frac{13}{9}$.
2. (a) $11 + 5i$. (b) $4 + i$. (c) $-2 + i$. (d) $12 + 0i$. (e) $2 - \frac{1}{2}i$.
(f) $0 + 2\pi i$. (g) $(3 + \sqrt{2}) + 3i$. (h) $(6\sqrt{2} - 3) + (1 + \sqrt{2})i$.
(i) $(\pi + \sqrt{3}) + (2 + \sqrt{2})i$.
3. (a) $-5 + 2i$. (b) $3\sqrt{2} - \frac{1}{2}i$. (c) $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$.
4. (a) $10 + 0i$. (b) $5 + 3i$. (c) $-1 + 2i$. (d) $\frac{5}{2} - i$. (e) $\frac{5}{2} + \frac{7}{2}i$.
(f) $12 + 0i$. (g) $-4\sqrt{2} + 4i$. (h) $(3\sqrt{2} - \pi) + (4 - \sqrt{2})i$.
(i) $\left(\dfrac{1}{\sqrt{3}} - 2\right) + (1 + \sqrt{2})i$.
8. (a) $-6 - 2i$. (b) $6 - \frac{5}{2}i$. (c) $(\sqrt{2} - 4) + (1 - \sqrt{2})i$.
(d) $6 + (\sqrt{2} - 6)i$.

EXERCISE 7.3 (page 160)

1. (a) -1. (b) i. (c) 1. (d) $-i$. (e) $-i$.
(f) 6. (g) 2. (h) -24. (i) 1. (j) -24.
2. (a) $22 + 29i$. (b) $0 + 30i$. (c) $\frac{27}{4} - \frac{7}{2}i$.
(d) $52 + 0i$. (e) $\frac{1}{4} - \frac{17}{4}i$. (f) $(-\pi^2 - 16) + 0i$.
(g) $(3\sqrt{2} - 2) + (3 + 2\sqrt{2})i$. (h) $-19\sqrt{2} + 9i$.
(i) $14 + 5i\sqrt{2}$. (j) $36 + 9i\sqrt{2}$.

7. $uv = 0 + 5i$, $u^2 + v^2 = 0 - 8i$, $u^3 - v^3 = -9 - 9i$.

8. -8. **9.** No. **10.** No.

12.

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

16. (a) $-1 + 0i$. (b) $-7 + 24i$. (c) $-\sqrt{2} - i\sqrt{2}$. (d) $69 - 13i$.

EXERCISE 7.4 (page 165)

1. (a) $4 + 3i$. (b) $-5 + 4i$. (c) $4 - 0i$. (d) $-3 + 3i$.
(e) $0 + 5i$. (f) $8 - 5i$.

2. (a) 8 and $-6i$. (b) -10 and $-8i$. (c) 8 and 0.
(d) -6 and $-6i$. (e) 0 and $-10i$. (f) 16 and $10i$.

3. Complex numbers $a + 0i$.

4. (a) 37. (b) 20. (c) $\frac{1}{2}$. (d) 16. (e) 7. (f) 52. (g) $\frac{1}{4}$. (h) 1.

7. (a) $4 + 4i$. (b) $-5 + 15i$. (c) $8 + 2i$.

8. (a) $4 + 7i$. (b) $-12 - 28i$. (c) $-15 + 23i$.

EXERCISE 7.5 (page 167)

1. (a) $\frac{34}{53} + \frac{13}{53}i$. (b) $-\frac{6}{5} - \frac{9}{10}i$. (c) $-\frac{4}{5} - \frac{1}{10}i$.
(d) $\frac{27}{25} - \frac{14}{25}i$. (e) $\frac{5}{13} - \frac{12}{13}i$. (f) $-\frac{11}{29} - \frac{13}{29}i$.
(g) $\dfrac{\pi^2 - 16}{\pi^2 + 16} + \dfrac{8\pi}{\pi^2 + 16}i$. (h) $\dfrac{2 + 3\sqrt{2}}{13} + \dfrac{3 - 2\sqrt{2}}{13}i$. (i) $-\frac{17}{11}\sqrt{2} - \frac{15}{11}i$.

2. (a) $-3i$ and $\frac{1}{3}i$. (b) $\frac{1}{5} + \frac{2}{5}i$ and $1 - 2i$.
(c) $\dfrac{4 + \sqrt{6}}{19} + \dfrac{4\sqrt{2} - \sqrt{3}}{19}i$ and $\dfrac{4 + \sqrt{6}}{3} + \dfrac{\sqrt{3} - 4\sqrt{2}}{3}i$.

3. (a) $\frac{4}{17} - \frac{1}{17}i$. (b) $\frac{3}{13} + \frac{2}{13}i$. (c) $\frac{2}{15} - \frac{1}{15}i$. (d) $-\frac{3}{20} - \frac{1}{20}i$.
(e) $1 + i$. (f) $\dfrac{\sqrt{3}}{4} - \dfrac{1}{4}i$. (g) $\frac{1}{3} + \frac{1}{6}i\sqrt{2}$. (h) $\dfrac{\sqrt{3}}{2} - \dfrac{i\sqrt{2}}{5}$.

4. $\dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2}i$.

5. (a) $\dfrac{1}{2} - \dfrac{5}{2}i$. (b) $\dfrac{-19}{13} + \dfrac{4}{13}i$. (c) $-\dfrac{9}{40} + \dfrac{17}{40}i$. (d) $\dfrac{8}{25} + \dfrac{6}{25}i$.

EXERCISE 7.6 (page 170)

1. (a) $0 + 5i$. (b) $4 - 11i$. (c) $0 + 10i$.
(d) $0 + 36i$. (e) $-9 + 0i$. (f) $0 + 60i$.
(g) $-2\sqrt{2} + 0i$. (h) $-13\sqrt{5} + 0i$. (i) $-14\sqrt{3} + 0i$.

2. (a) $\frac{1}{12} \pm i\sqrt{47}$. (b) $\dfrac{\sqrt{2}}{6} \pm i\dfrac{\sqrt{10}}{6}$. (c) $\dfrac{5\sqrt{2}}{4} \pm i\dfrac{\sqrt{46}}{4}$. (d) $\dfrac{\sqrt{3}}{6} \pm \dfrac{i\sqrt{5}}{2}$.
(e) $0 + \left(\dfrac{1 \pm \sqrt{17}}{2}\right)i$. (f) $\pm\dfrac{\sqrt{2}}{3} - \dfrac{i}{3}$.
(g) $0 + i(-3 \pm 2\sqrt{3})$. (h) $\pm\dfrac{\sqrt{23}}{4} + \dfrac{i}{4}$.

4. (a) $-\frac{5}{2} \pm \frac{\sqrt{7}}{2}i$. (b) $1 \pm \frac{i\sqrt{2}}{2}$. (c) $\frac{5}{12} \pm \frac{i\sqrt{23}}{12}$. (d) $-\frac{1}{6} \pm \frac{i\sqrt{23}}{6}$.
7. $1 + i$ and $2 - i$.
10. (a) $x^2 - 4x + 5 = 0$. (b) $x^2 + 6x + 13 = 0$. (c) $x^2 + 1 = 0$.
(d) $x^2 + (2\sqrt{2})x + 11 = 0$. (e) $x^2 - 12x + 39 = 0$. (f) $x^2 - x + \frac{3}{4} = 0$.

EXERCISE 7.7 (page 175)

1. (a) $4 + 0i$. (b) $-10 + 0i$. (c) $0 + 3i$. (d) $0 - 5i$.
(e) $1 + 0i$. (f) $-3 + 0i$. (g) $0 + 5i$. (h) $\sqrt{3} + i$.
(i) $-\frac{3}{2}\sqrt{2} + (\frac{3}{2}\sqrt{2})i$. (j) $\frac{5}{2} + i\left(\frac{5\sqrt{3}}{2}\right)$. (k) $7 + 0i$.
(l) $2 - 2i\sqrt{3}$. (m) $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. (n) $-\frac{3}{2} - \frac{3i\sqrt{3}}{2}$.
(o) $-\sqrt{3} - i$. (p) $\frac{3}{2}\sqrt{3} + \frac{3}{2}i$. (q) -1.
(r) $-\frac{3}{2}\sqrt{3} + \frac{3}{2}i$. (s) $-3\sqrt{3} - 3i$.

2. (a) $4 \text{ cis } 0°$. (b) $3 \text{ cis } \frac{\pi}{2}$. (c) $2 \text{ cis } \pi$. (d) $\text{cis } 270°$.
(e) $2 \text{ cis } 60°$. (f) $2 \text{ cis } 300°$. (g) $2 \text{ cis } 120°$. (h) $2 \text{ cis } 240°$.
(i) $6\sqrt{2} \text{ cis } 45°$. (j) $4 \text{ cis } 150°$. (k) $5 \text{ cis } 270°$. (l) $12 \text{ cis } 240°$.
(m) $6 \text{ cis } 120°$. (n) $4\sqrt{2} \text{ cis } 225°$. (o) $10 \text{ cis } 0°$. (p) $\text{cis } 120°$.
(q) $\sqrt{2} \text{ cis } \pi$. (r) $2 \text{ cis } 330°$.
3. (a) $4 \text{ cis } 2k\pi, k \in I$. (b) $2 \text{ cis } (2k + 1)\pi, k \in I$.
(c) $\text{cis}(\frac{1}{2} + 2k)\pi, k \in I$. (d) $6 \text{ cis } (\frac{3}{2} + 2k)\pi, k \in I$.
(e) $2 \text{ cis } (\frac{5}{6} + 2k)\pi, k \in I$. (f) $6 \text{ cis } (\frac{7}{4} + 2k)\pi$.
4. (a) $1.532 + 1.2856i$. (b) $-.5211 + 2.9544i$. (c) $-.9397 - .3420i$.
(d) $-2.9544 - .5211i$. (e) $.1744 - 1.9924i$. (f) $-1.6648 + 3.6372i$.
5. (a) $2k\pi, k \in I$. (b) $(2k + 1)\pi, k \in I$. (c) $(2k + \frac{1}{2})\pi, k \in I$.
(d) $(2k + \frac{3}{2})\pi, k \in I$.
8. (a) 6. (b) $6 \text{ cis } \frac{3\pi}{2}$. (c) $6 \text{ cis } \pi$. (d) $6 \text{ cis } \frac{\pi}{2}$.
(e) $\sqrt{2} \text{ cis } \frac{7\pi}{4}$. (f) $3\sqrt{2} \text{ cis } \frac{\pi}{4}$. (g) $2 \text{ cis } \frac{\pi}{6}$. (h) $2 \text{ cis } \frac{\pi}{3}$.

EXERCISE 7.8 (page 177)

1. (a) $3\sqrt{2} + 3i\sqrt{2}$. (b) $0 + 8i$. (c) $0 - 4i$.
(d) $-1 + 0i$. (e) $\sqrt{2} - i\sqrt{2}$. (f) $\frac{2}{3}\sqrt{2} + \frac{2i\sqrt{2}}{3}$.
(g) $0 + \frac{1}{2}i$. (h) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. (i) $-6 + 6i\sqrt{3}$.
(j) $-\frac{2}{5} + 0i$. (k) $-6\sqrt{2} + 6i\sqrt{2}$. (l) $-5\sqrt{2} - 5i\sqrt{2}$.
3. (a) $\frac{9}{2} + \frac{9i\sqrt{3}}{2}$. (b) $0 + 2i$. (c) $-\frac{9}{2} + \frac{9i\sqrt{3}}{2}$.
5. (a) $\frac{1}{2} \text{ cis } 300°$. (b) $2 \text{ cis } 30°$. (c) $\frac{\sqrt{2}}{2} \text{ cis } 225°$.
6. (a) $2 + 2i\sqrt{3}$. (b) $-4 + 0i$.

EXERCISE 7.9 (page 180)

1. (a) $\frac{1}{2}+\frac{\sqrt{3}}{2}i.$ (b) $1+0i.$ (c) $-1+0i.$
(d) $\frac{1}{2}-\frac{\sqrt{3}}{2}i.$ (e) $0+4i.$ (f) $-\frac{81}{2}+\frac{81i\sqrt{3}}{2}.$
(g) $\frac{1}{8}+0i.$ (h) $\frac{\sqrt{3}}{8}-\frac{1}{8}i.$ (i) $243+0i.$
(j) $-\frac{1}{9}+0i.$ (k) $16-16i.$ (l) $16\sqrt{3}+16i.$
(m) $0+512i.$ (n) $0-i.$ (o) $\frac{1}{16}+0i.$
(p) $-1+0i.$ (q) $-64\sqrt{3}-64i.$ (r) $65{,}536+0i.$

2. (a) $64+0i.$ (b) $\frac{1}{2048}-\frac{\sqrt{3}}{2048}i.$ (c) $0-\frac{1}{2}i.$ (d) $-32{,}768+0i.$

3. (a) $\cos 2\theta=\cos^2\theta-\sin^2\theta,$ $\sin 2\theta = 2\sin\theta\cos\theta.$
(b) $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta,$ $\sin 4\theta = 4\sin\theta\cos\theta\,(\cos^2\theta-\sin^2\theta).$
(c) $\cos 5\theta=\cos\theta\,(\cos^4\theta-10\cos^2\theta\sin^2\theta+5\sin^4\theta),$
$\sin 5\theta=\sin\theta(5\cos^4\theta-10\cos^2\theta\sin^2\theta+\sin^4\theta).$
(d) $\cos 6\theta=\cos^6\theta-15\cos^4\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta,$
$\sin 6\theta=6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta.$

EXERCISE 7.10 (page 183)

1. (a) $\pm\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i\right).$ (b) $\pm(\sqrt{2}-i\sqrt{2}).$ (c) $\pm(2+2i\sqrt{3}).$
(d) $1+i\sqrt{3},\ 1-i\sqrt{3},\ -2+0i.$ (e) $3+0i,\ -\frac{3}{2}+\frac{3\sqrt{3}}{2}i,\ -\frac{3}{2}-\frac{3\sqrt{3}}{2}i.$
(f) $0+2i,\ \sqrt{3}-i,\ -1-i\sqrt{3}.$
(g) $\sqrt[3]{9}$ cis $(30+60k)^\circ,\ k=0, 1, 2, 3, 4, 5.$
(h) 2 cis $(30+72k)^\circ,\ k=0, 1, 2, 3, 4.$
(i) cis $60k^\circ,\ k=0, 1, 2, 3, 4, 5.$ (j) $2+2i,\ -2+2i,\ -2-2i,\ 2-2i.$
(k) $\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1+0i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i.$
(l) $\sqrt{3}+i,\ -1+i\sqrt{3},\ -\sqrt{3}-i,\ 1-i\sqrt{3}.$
(m) cis $(15+60k)^\circ,\ k=0, 1, 2, 3, 4, 5.$
(n) $\sqrt{2}$ cis $(27+72k)^\circ,\ k, =0, 1, 2, 3, 4.$
(o) $\pm(\sqrt{\frac{6}{2}}-i\sqrt{\frac{2}{2}}).$
(p) cis $(15+45k)^\circ,\ k=0, 1, 2, 3, 4, 5, 6, 7.$

2. (a) $4+0i,\ -2+2i\sqrt{3},\ -2-2i\sqrt{3}.$ (b) $\pm(2\sqrt{2}+2i\sqrt{2}).$
(c) $\pm\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i\right),\ \pm\left(\frac{-3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i\right).$
(d) 2 cis $72k^\circ,\ k=0, 1, 2, 3, 4.$ (e) cis $(36+72k)^\circ,\ k=0, 1, 2, 3, 4.$
(f) $\pm\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i\right).$ (g) $\sqrt[4]{8}$ cis $(60+90k)^\circ,\ k=0, 1, 2, 3.$
(h) $\sqrt[6]{5}$ cis $\left(\frac{\theta}{3}+120k\right)^\circ,\ k=0, 1, 2,$ where $\theta=\tan^{-1}\frac{1}{2},\ 0^\circ<\theta<90^\circ.$

3. (a) $\pm(2+i).$ (b) $\pm(3-2i).$ **4.** $4.175+1.397i,\ .825-.3968i.$

REVIEW EXERCISE 7 (page 185)

1. (a) 3 and 5. (b) 4 and -1. (c) $-\frac{1}{2}$ and 2. (d) 5 and 0.

2. (a) $\sqrt{34}$. (b) $\sqrt{17}$. (c) $\frac{1}{2}\sqrt{17}$. (d) 5.

3. (a) $-1+3i$. (b) $-1+3i$. (c) $7-7i$. (d) $-7+7i$.

4. (a) $-5-2i$. (b) $-5-2i$. (c) $-3-4i$. **5.** (a) $-5+9i$. (b) $1-i$.

7. (a) $-17-i$. (b) $-17-i$. (c) $65+55i$. (d) $65+55i$.
(e) $-31-24i$. (f) $-31-24i$. (g) $1-18i$. (h) $1-18i$.

8. (a) 10. (b) -6. (c) -8. (d) 29. (e) 10. (f) 25. (g) $2-3i$.
(h) $2-3i$. (i) $-17+i$. (j) $-17+i$. (k) $65-55i$. (l) $65-55i$.

9. (a) $-\frac{13}{10}-\frac{11}{10}i$. (b) $\frac{9}{25}-\frac{13}{25}i$. (c) $-\frac{26}{29}-\frac{7}{29}i$.
(d) 1. (e) $\frac{5}{29}-\frac{2}{29}i$. (f) $-\frac{3}{10}-\frac{1}{10}i$.
(g) $-\frac{4}{25}+\frac{3}{25}i$. (h) $\frac{13}{1450}-\frac{11}{1450}i$.

10. (a) $\frac{14}{25}+\frac{23}{25}i$. (b) $-\frac{18}{13}-\frac{12}{13}i$.

11. (a) 4. (b) $-\dfrac{3\sqrt{2}}{2}-\dfrac{3i\sqrt{2}}{2}$. (c) $-\sqrt{3}-i$. (d) 4.

12. (a) 6 cis 0°. (b) cis 240°. (c) 6 cis 150°. (d) 2 cis 30°.

13. (a) $0+15i$. (b) $\sqrt{2}+i\sqrt{2}$. (c) $2+2i\sqrt{3}$. (d) $0+3i\sqrt{3}$.
(e) $\dfrac{\sqrt{3}}{4}-\dfrac{1}{4}i$. (f) $\dfrac{3\sqrt{2}}{8}-\dfrac{3i\sqrt{2}}{8}$.

14. (a) $16-16i$. (b) $-128-128i\sqrt{3}$.

15. (a) $\frac{3}{2}\sqrt{3}+\frac{3}{2}i,\ -\frac{3}{2}\sqrt{3}+\frac{3}{2}i,\ 0-3i$.
(b) $\sqrt[5]{16}\ \text{cis}\ (-12+72k)^\circ,\ k=0, 1, 2, 3, 4$.

16. (a) $3\ \text{cis}\ (67\frac{1}{2}+90k)^\circ,\ k=0, 1, 2, 3$.
(b) $\sqrt[4]{8}\ \text{cis}\ (-15+90k)^\circ,\ k=0, 1, 2, 3$.

Chapter 8

EXERCISE 8.2 (page 189)

1. $\begin{pmatrix}A & B\\ A & B\end{pmatrix}$ and $\begin{pmatrix}A & B\\ B & A\end{pmatrix}$.

2. $\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\right\}$.

3.
$$\begin{gathered}\left\{\begin{pmatrix}ABCD\\ABCD\end{pmatrix}, \begin{pmatrix}ABCD\\ABDC\end{pmatrix}, \begin{pmatrix}ABCD\\ACBD\end{pmatrix}, \begin{pmatrix}ABCD\\ACDB\end{pmatrix}, \begin{pmatrix}ABCD\\ADBC\end{pmatrix}, \begin{pmatrix}ABCD\\ADCB\end{pmatrix},\right.\\ \begin{pmatrix}ABCD\\BACD\end{pmatrix}, \begin{pmatrix}ABCD\\BADC\end{pmatrix}, \begin{pmatrix}ABCD\\BCAD\end{pmatrix}, \begin{pmatrix}ABCD\\BCDA\end{pmatrix}, \begin{pmatrix}ABCD\\BDAC\end{pmatrix}, \begin{pmatrix}ABCD\\BDCA\end{pmatrix},\\ \begin{pmatrix}ABCD\\CABD\end{pmatrix}, \begin{pmatrix}ABCD\\CADB\end{pmatrix}, \begin{pmatrix}ABCD\\CBAD\end{pmatrix}, \begin{pmatrix}ABCD\\CBDA\end{pmatrix}, \begin{pmatrix}ABCD\\CDAB\end{pmatrix}, \begin{pmatrix}ABCD\\CDBA\end{pmatrix},\\ \left.\begin{pmatrix}ABCD\\DABC\end{pmatrix}, \begin{pmatrix}ABCD\\DACB\end{pmatrix}, \begin{pmatrix}ABCD\\DBAC\end{pmatrix}, \begin{pmatrix}ABCD\\DBCA\end{pmatrix}, \begin{pmatrix}ABCD\\DCAB\end{pmatrix}, \begin{pmatrix}ABCD\\DCBA\end{pmatrix}\right\}.\end{gathered}$$

4. $\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$.

5. (a) $\begin{pmatrix}A&B&C&D&E\\A&B&C&D&E\end{pmatrix}$, $\begin{pmatrix}A&B&C&D&E\\A&B&D&C&E\end{pmatrix}$, $\begin{pmatrix}A&B&C&D&E\\A&C&B&D&E\end{pmatrix}$,
$\begin{pmatrix}A&B&C&D&E\\A&C&D&B&E\end{pmatrix}$, $\begin{pmatrix}A&B&C&D&E\\A&D&B&C&E\end{pmatrix}$, $\begin{pmatrix}A&B&C&D&E\\A&D&C&B&E\end{pmatrix}$.

(b) Same as (a) but with A and E interchanged on the second line of each symbol.

(c) $\begin{pmatrix}A&B&C&D&E\\A&B&C&D&E\end{pmatrix}$ and $\begin{pmatrix}A&B&C&D&E\\E&B&C&D&A\end{pmatrix}$.

(d) $\begin{pmatrix}A&B&C&D&E\\A&C&D&B&E\end{pmatrix}$ and $\begin{pmatrix}A&B&C&D&E\\E&C&D&B&A\end{pmatrix}$.

6. (a) $\begin{pmatrix}1&2&3&4&5\\1&2&3&4&5\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4&5\\1&2&5&4&3\end{pmatrix}$.

(b) $\begin{pmatrix}1&2&3&4&5\\a&b&3&c&5\end{pmatrix}$ with abc replaced by 1 2 4, 1 4 2, 2 1 4, 2 4 1, 4 1 2, 4 2 1.

(c) $\begin{pmatrix}1&2&3&4&5\\a&b&c&5&4\end{pmatrix}$ with abc replaced by 1 2 3, 1 3 2, 2 1 3, 2 3 1, 3 1 2, 3 2 1.

(d) $\begin{pmatrix}1&2&3&4&5\\3&5&1&4&2\end{pmatrix}$. (e) $\begin{pmatrix}1&2&3&4&5\\3&2&5&4&1\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4&5\\3&4&5&2&1\end{pmatrix}$.

7. (a) $\begin{pmatrix}1&2&3&4&5&6\\a&b&c&4&5&6\end{pmatrix}$ with abc replaced by 1 2 3, 1 3 2, 2 1 3, 2 3 1, 3 1 2, 3 2 1.

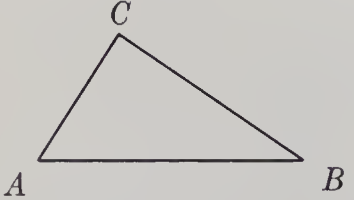
(b) $\begin{pmatrix}1&2&3&4&5&6\\1&2&3&a&b&c\end{pmatrix}$ with abc replaced by 4 5 6, 4 6 5, 5 4 6, 5 6 4, 6 4 5, 6 5 4.

(c) $\begin{pmatrix}1&2&3&4&5&6\\4&5&6&1&2&3\end{pmatrix}$. (d) $\begin{pmatrix}1&2&3&4&5&6\\6&5&3&4&2&1\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix}$.

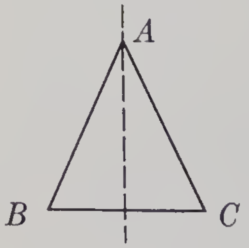
8. $(n-2)!$.

EXERCISE 8.3 (page 191)

1. Only the identity. $\begin{pmatrix}A&B&C\\A&B&C\end{pmatrix}$.



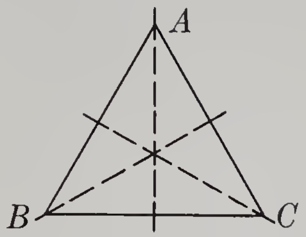
2. Identity and rotation of 180° about the median from A.



$\begin{pmatrix}A&B&C\\A&B&C\end{pmatrix}$ and $\begin{pmatrix}A&B&C\\A&C&B\end{pmatrix}$.

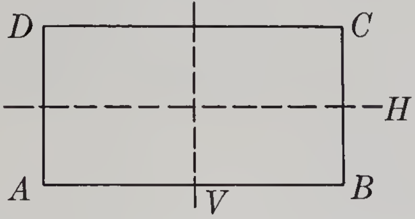
3. Only the identity. $\begin{pmatrix}A&B&C&D\\A&B&C&D\end{pmatrix}$.

4. Identity; rotations of 120° and 240° about the centre; rotations of 180° about the three medians.



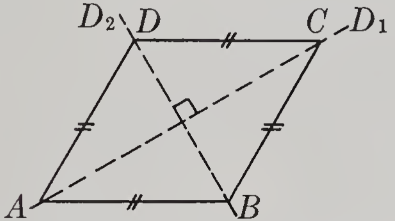
$$\begin{pmatrix}A & B & C\\ A & B & C\end{pmatrix}, \begin{pmatrix}A & B & C\\ C & A & B\end{pmatrix}, \begin{pmatrix}A & B & C\\ B & C & A\end{pmatrix},$$
$$\begin{pmatrix}A & B & C\\ A & C & B\end{pmatrix}, \begin{pmatrix}A & B & C\\ C & B & A\end{pmatrix}, \begin{pmatrix}A & B & C\\ B & A & C\end{pmatrix}.$$

5. Identity; rotation of 180° about centre; rotations of 180° about lines H and V.



$$\begin{pmatrix}A & B & C & D\\ A & B & C & D\end{pmatrix}, \begin{pmatrix}A & B & C & D\\ C & D & A & B\end{pmatrix},$$
$$\begin{pmatrix}A & B & C & D\\ D & C & B & A\end{pmatrix}, \begin{pmatrix}A & B & C & D\\ B & A & D & C\end{pmatrix}.$$

6. Identity; rotation of 180° about centre; rotations of 180° about lines D_1 and D_2.

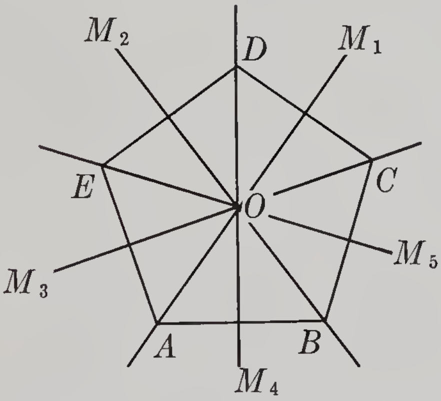


$$\begin{pmatrix}A & B & C & D\\ A & B & C & D\end{pmatrix}, \begin{pmatrix}A & B & C & D\\ C & D & A & B\end{pmatrix},$$
$$\begin{pmatrix}A & B & C & D\\ A & D & C & B\end{pmatrix}, \begin{pmatrix}A & B & C & D\\ C & D & A & B\end{pmatrix}.$$

7. The motions have been listed in the text.

$$R_0 = \begin{pmatrix}A & B & C & D\\ A & B & C & D\end{pmatrix}, R_{90} = \begin{pmatrix}A & B & C & D\\ B & C & D & A\end{pmatrix}, R_{180} = \begin{pmatrix}A & B & C & D\\ C & D & A & B\end{pmatrix}, R_{270} = \begin{pmatrix}A & B & C & D\\ D & A & B & C\end{pmatrix}$$
$$H = \begin{pmatrix}A & B & C & D\\ D & C & B & A\end{pmatrix}, V = \begin{pmatrix}A & B & C & D\\ B & A & D & C\end{pmatrix}, D_1 = \begin{pmatrix}A & B & C & D\\ A & D & C & B\end{pmatrix}, D_2 = \begin{pmatrix}A & B & C & D\\ C & B & A & D\end{pmatrix}.$$

8. Identity; rotations of 72°, 144°, 216°, 288° about O; rotations of 180° about M_1, M_2, M_3, M_4, M_5.



$$R_0 = \begin{pmatrix}A & B & C & D & E\\ A & B & C & D & E\end{pmatrix}, R_{72} = \begin{pmatrix}A & B & C & D & E\\ B & C & D & E & A\end{pmatrix},$$
$$R_{144} = \begin{pmatrix}A & B & C & D & E\\ C & D & E & A & B\end{pmatrix}, R_{216} = \begin{pmatrix}A & B & C & D & E\\ D & E & A & B & C\end{pmatrix},$$
$$R_{288} = \begin{pmatrix}A & B & C & D & E\\ E & A & B & C & D\end{pmatrix}, M_1 = \begin{pmatrix}A & B & C & D & E\\ A & E & D & C & B\end{pmatrix},$$
$$M_2 = \begin{pmatrix}A & B & C & D & E\\ C & B & A & E & D\end{pmatrix}, M_3 = \begin{pmatrix}A & B & C & D & E\\ E & D & C & B & A\end{pmatrix},$$
$$M_4 = \begin{pmatrix}A & B & C & D & E\\ B & A & E & D & C\end{pmatrix}, M_5 = \begin{pmatrix}A & B & C & D & E\\ D & C & B & A & E\end{pmatrix}.$$

9. Identity; rotations of 60°, 120°, 180°, 240°, 300° about the centre; rotations of 180° about each of the three diagonals joining opposite vertices; rotations of 180° about each of the three lines joining midpoints of opposite sides. Altogether there are 12 such motions.

10. Identity; rotations of 45°, 90°, 135°, 180°, 225°, 270°, 315° about the centre; rotations of 180° about each of the four diagonals joining opposite vertices; rotations of 180° about each of the four joining midpoints of opposite sides. Altogether there are 16 such motions.

11. $2n$.

12. The sixteen permutations listed in the answer to question (3) of Exercise 8.1 which do not appear in the answer to question (7) of this Exercise.

EXERCISE 8.4 (page 194)

1. (a) $(x, y) \to (x + 3, y + 4)$ or $T_{3,4}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3 \\ y + 4 \end{bmatrix}$.

(b) $(x, y) \to (x - 2, y + 1)$. (c) $(x, y) \to (x, y)$.

(d) $(x, y) \to (x, y - 3)$. (e) $(x, y) \to (x - 3, y - 3)$.

(f) $(x, y) \to \left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)$ or $R_{(\pi/3)}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x - \sqrt{3}y) \\ \frac{1}{2}(\sqrt{3}x + y) \end{bmatrix}$.

(g) $(x, y) \to (-y, x)$. (h) $(x, y) \to (-x, -y)$.

(i) $(x, y) \to \left(\frac{x + y}{\sqrt{2}}, \frac{-x + y}{\sqrt{2}}\right)$. (j) $(x, y) \to (y, -x)$.

2. (a) $T_{-3,-2}$. (b) $T_{5,-2}$. (c) $T_{4,0}$ (d) $T_{-2,3}$. (e) $T_{3,3}$.

3. $T_{-3,0}$; $T_{0,-2}$; $T_{3,0}$; $T_{0,2}$.

4. (a) $R_{(-\pi/3)}$. (b) $R_{(\pi/3)}$. (c) $R_{(-\pi/3)}$. (d) $R_{(\pi/3)}$.

5. (a) $R_{(\pi/3)}$. (b) $R_{(-\pi/3)}$. (c) $R_{(\pi/3)}$. (d) $R_{(-\pi/3)}$.

6. $f_{2,0}$; $f_{-2,0}$; $f_{0,2}$; $f_{0,-2}$.
$x \to 2x$ $x \to -2x$ $x \to 2$ $x \to -2$

7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, etc. 29 matrices.

EXERCISE 8.5 (page 197)

1.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
(a)	3	-3	-3	0	-6	7
(b)	29	29	45	50	850	205
(c)	49	49	81	100	2704	961
(d)	16	19	24	25	47	51
(e)	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{1}{2}$	1	$\frac{2}{15}$	9
(f)	5	5	6	5	5	6
(g)	10	10	6	5	30	6
(h)	5	2	3	5	2	6
(i)	2	5	6	5	3	3
(j)	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	5	$\frac{13}{4}$	$\frac{17}{4}$
(k)	25	32	729	3125	32^3	6^8
(l)	-5	-8	-15	-20	16	30
(m)	1	2	3	0	2	0
(n)	2	1	0	0	0	1

2.

	$A + B = B + A$	AB	BA
(a)	$\begin{bmatrix} 4 & 0 \\ 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 4 & 6 \\ 8 & -24 \end{bmatrix}$	$\begin{bmatrix} 8 & 5 \\ -16 & -28 \end{bmatrix}$
(b)	$\begin{bmatrix} 5 & -4 \\ 9 & 5 \end{bmatrix}$	$\begin{bmatrix} -21 & -14 \\ 26 & 10 \end{bmatrix}$	$\begin{bmatrix} 8 & -18 \\ 17 & -19 \end{bmatrix}$
(c)	$\begin{bmatrix} 5 & 0 \\ 8 & 1 \end{bmatrix}$	$\begin{bmatrix} 9 & -4 \\ 8 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 20 & 5 \end{bmatrix}$

(d)	$\begin{bmatrix} -1 & 2 \\ 8 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 4 \\ -4 & 5 \end{bmatrix}$	$\begin{bmatrix} -3 & -2 \\ 16 & 7 \end{bmatrix}$
(e)	$\begin{bmatrix} 7 & 4 \\ 9 & 1 \end{bmatrix}$	$\begin{bmatrix} 20 & 8 \\ 19 & 10 \end{bmatrix}$	$\begin{bmatrix} 22 & 8 \\ 16 & 8 \end{bmatrix}$
(f)	$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
(g)	$\begin{bmatrix} \frac{5}{2} & 0 \\ 0 & \frac{17}{4} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

EXERCISE 8.6 (page 202)

1.

	$P\theta Q$	$Q\theta P$	P^2	Q^2
(a)	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
(b)	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$
(c)	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.
(d)	$\begin{pmatrix} a & b & c & d & e \\ d & a & e & b & c \end{pmatrix}$	$\begin{pmatrix} a & b & c & d & e \\ e & d & a & b & c \end{pmatrix}$	$\begin{pmatrix} a & b & c & d & e \\ d & b & a & c & e \end{pmatrix}$	$\begin{pmatrix} a & b & c & d & e \\ c & d & e & a & b \end{pmatrix}$.
(e)	$\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 3&5&2&7&6&8&9&4&1 \end{pmatrix}$	$\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 8&1&9&2&4&3&6&7&5 \end{pmatrix}$	$\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 3&4&1&2&6&9&5&7&8 \end{pmatrix}$	$\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 1&2&8&7&5&3&6&4&9 \end{pmatrix}$.

2. (a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} a & b & c & d & e \\ e & b & d & c & a \end{pmatrix}$.

3. Multiplication of permutations is associative.

4. (a) E_2. (b) R_{120}. (c) E_3.

5. (a) R_{270}. (b) R_{270}. (c) R_{180}. (d) R_0. (e) R_{180}. (f) R_{180}.
(g) H. (h) V. (i) R_{180}. (j) R_{180}. (k) R_{90}. (l) D_2.

6.

θ	I	R_x	R_y	R_{xy}
I	I	R_x	R_y	R_{xy}
R_x	R_x	I	R_{xy}	R_y
R_y	R_y	R_{xy}	I	R_x
R_{xy}	R_{xy}	R_y	R_x	I

EXERCISE 8.7 (page 206)

1. (a) $T_{4,5}$. (b) $T_{8,0}$. (c) $T_{0,0}$. (d) $T_{-3,1}$.

3.

	$f\theta g$	$g\theta f$
(a)	$x \to 6x - 1$	$x \to 6x + 17$
(b)	$x \to 18x + 1$	$x \to 18x - 23$
(c)	$x \to 9x^2 - 12x + 7$	$x \to 3x^2 + 7$
(d)	$x \to x^4 + 2x^3 + 2x^2 + x + 1$	$x \to x^4 - 2x^3 + 4x^2 - 3x + 3$

EXERCISE 8.8 (page 208)

The following are groupoids.
1 (a), (b), (c); 2 (a), (b), (e), (f); 3 (a), (c); 4 (a), (b), (c), (e), (f); 5 (a), (b), (c), (d); 6 (a), (b), (c), (d), (e); 7 (b), (c), (d), (f), (g), (h), (i), (j), (l), (n); 8 (a), (b), (c), (d), (e); 9 (a), (b); 10 (a), (c), (e); 11 (a), (b), (c); 12 (a), (b), (c); 13 (a), (b), (c), (d); 14; 15.

EXERCISE 8.9 (page 213)

The following groupoids of Exercise 8.8 are semigroups.
1 (a), (b); 2 (a), (b), (e), (f); 3 (a), (c); 4 (a), (b), (e); 5 (a), (b), (c), (d); 6 (a), (b), (c), (d), (e); 7 (f), (g), (h), (i), (j), (n); 8 (a), (b), (c), (d) (e); 9 (a), (b); 10 (a), (c), (e); 11 (a), (b), (c); 12 (a), (b), (c); 13 (a), (c), (d); 14; 15.

EXERCISE 8.10 (page 215)

The following algebraic systems in Exercise 8.8 have

(a) no identities:
    2 (b), (c), (d); 4 (b), (d); 7 (a), (b), (c), (d), (h), (k), (m), (n), (o), (p), (q); 10 (b), (d).

(b) right but not left identities:
    1 (c); 4 (c), (f); 6 (f); 7 (e), (i), (l).

(c) left but not right identities:
    7 (j); 13 (c).

(d) both left and right identities:
    1 (a), (b); 2 (a), (e) (f); 3 (a), (b), (c); 4 (a), (e); 5 (a), (b), (c), (d); 6 (a), (b), (c), (d), (e); 7 (f), (g); 8 (a), (b), (c), (d), (e); 9 (a), (b); 10 (a), (c), (e); 11 (a), (b), (c); 12 (a), (b), (c); 13 (a), (b), (d); 14; 15.

EXERCISE 8.11 (page 220)

1. The following algebraic systems of Exercise 8.8 with identities (see Answers for (1d) of Exercise 8.10) have inverses for *all* elements.
1 (a); 2 (a); 4 (a), (e); 5 (a), (c);
6 (a) [1 and -1 are their own inverses; i and $-i$ are inverses of each other];
6 (b), (d), (e);
8 (a) each element is its own inverse;
(b) each element is its own inverse;
(c) R_0, R_{180}, D_1, D_2, H, V are their own inverses;
R_{90} and R_{270} are inverses of each other;
(d) each element is its own inverse;
(e) R_0, E_1, E_2, E_3 are their own inverses; R_{120} and R_{240} are inverses.
9 (a); 10 (c); (e) f_1, f_2, f_3, f_6 are their own inverses; f_4 and f_5 are inverses;
11 (a), (b), (c); 12 (a), (b), (c);
13 (a) A is its own inverse; B and C are inverses;
(b) each element is its own inverse;
(d) each element is its own inverse.

2. (a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. (b) $\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{C} & \mathrm{D} & \mathrm{B} & \mathrm{A} \end{pmatrix}$. (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$.
(d) $\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \\ \mathrm{B} & \mathrm{E} & \mathrm{A} & \mathrm{C} & \mathrm{D} \end{pmatrix}$. (e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$.

3. (a) $f^{-1} : x \longrightarrow \frac{1}{4}x + \frac{1}{4}$. (b) $f^{-1} : x \longrightarrow 2x - 6$.
(c) $f^{-1} : x \longrightarrow -\frac{1}{2}x + \frac{3}{2}$. (d) $f^{-1} : x \longrightarrow -\frac{1}{4}x - \frac{7}{4}$.

EXERCISE 8.12 (page 224)

1. The following systems of Exercise 8.8 are groups.
   1 (a); 2 (a); 4 (a), (e); 5 (a), (c); 6 (a), (b), (d), (e); 8 (a), (b), (c) (d), (e); 9 (a); 10 (c), (e); 11 (a), (b), (c); 12 (a), (b), (c); 13 (a), (d).
2. The following systems of Exercise 8.8 are commutative groups.
   1 (a); 2 (a); 4 (a), (e); 5 (a), (c); 6 (a), (b), (d), (e); 8 (a), (b), (d); 9 (a); 10 (c); 11 (a), (b), (c); 13 (a), (d).
3. (b) and (d) are groups.
4. Rhombus.

EXERCISE 8.13 (page 226)

1. (a) $AB = \begin{bmatrix} 7 & 2 \\ 14 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$.
   (b) $AB = \begin{bmatrix} -9 & 3 \\ -2 & 2 \end{bmatrix}$, $BA = \begin{bmatrix} -4 & 24 \\ 1 & -3 \end{bmatrix}$.
   (c) $AB = \begin{bmatrix} 10 & -2 \\ 9 & 19 \end{bmatrix}$, $BA = \begin{bmatrix} 13 & 0 \\ 14 & 16 \end{bmatrix}$.
   (d) $AB = \begin{bmatrix} 15 & -6 \\ 9 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} -5 & 13 \\ 2 & 11 \end{bmatrix}$.
2. $(AB)C = A(BC)$
   (a) $\begin{bmatrix} 3 & 29 \\ 6 & 58 \end{bmatrix}$. (b) $\begin{bmatrix} -15 & -15 \\ -6 & 2 \end{bmatrix}$. (c) $\begin{bmatrix} 14 & 22 \\ -29 & 103 \end{bmatrix}$. (d) $\begin{bmatrix} 27 & 21 \\ 27 & -9 \end{bmatrix}$.
3. (a) A^{-1} does not exist; $B^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix}$.
   (b) $A^{-1} = \begin{bmatrix} -1 & \frac{5}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$; $B^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix}$.
   (c) $A^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{1}{13} \\ -\frac{1}{13} & \frac{4}{13} \end{bmatrix}$; $B^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{16} \\ -\frac{1}{8} & \frac{3}{16} \end{bmatrix}$.
   (d) $A^{-1} = \begin{bmatrix} \frac{2}{3} & -1 \\ \frac{1}{3} & 0 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} \frac{2}{27} & -\frac{5}{27} \\ \frac{5}{27} & -\frac{1}{27} \end{bmatrix}$.
7. No; the inverse of such a matrix does not necessarily have integral components.
11.

	I	R_x	R_y	R_{xy}
I	I	R_x	R_y	R_{xy}
R_x	R_x	I	R_{xy}	R_y
R_y	R_y	R_{xy}	I	R_x
R_{xy}	R_{xy}	R_y	R_x	I

EXERCISE 8.14 (page 230)

1. The following systems are rings. (a), (b), (c), (g), (i), (k), (l), (m).
2. Yes. **3.** Yes. **4.** (a) Yes. (b) Yes. (c) Yes. (d) No. (e) No. **5.** Yes.
6. (a) Commutative rings are 1 (a), (b), (c), (g), (i), (k), (l), (m); 2; 4 (b), (c); 5.
   (b) Rings with units are 1 (c), (g), (i), (k), (m); 2; 3; 4 (a), (b), (c); 5.
   (c) Commutative rings with unit are 1 (c), (g), (i), (m); 2; 4 (b), (c); 5.
7. In 3, 4 (a), (b). **9.** Example 4; also (5) of the Exercise.

EXERCISE 8.15 (page 233)

1. The following systems of Exercise 8.14 are fields. 1(c), (g), (i), (k), (m); 4(b); (5).

2. No. **4.** No.

EXERCISE 8.16 (page 237)

1. (a) $\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&2&1&8&7&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&7&8&1&2&6&5\end{pmatrix}, I.$

(b) $\begin{pmatrix}1&2&3&4&5&6&7&8\\2&3&4&1&6&7&8&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\3&4&1&2&7&8&5&6\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\4&1&2&3&8&5&6&7\end{pmatrix}, I.$

(c) $\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&6&2&4&8&7&3\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\1&4&8&5&2&3&7&6\end{pmatrix}, I.$

(d) $\begin{pmatrix}1&2&3&4&5&6&7&8\\3&2&6&7&4&1&5&8\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\6&2&1&5&7&3&4&8\end{pmatrix}, I.$

(e) $\begin{pmatrix}1&2&3&4&5&6&7&8\\8&4&3&7&5&1&2&6\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\6&7&3&2&5&8&4&1\end{pmatrix}, I.$ (f) $\begin{pmatrix}1&2&3&4&5&6&7&8\\5&8&7&6&1&4&3&2\end{pmatrix}, I.$

(g) $\begin{pmatrix}1&2&3&4&5&6&7&8\\2&1&5&6&3&4&8&7\end{pmatrix}, I.$ (h) $\begin{pmatrix}1&2&3&4&5&6&7&8\\7&8&4&3&6&5&1&2\end{pmatrix}, I.$

(i) $\begin{pmatrix}1&2&3&4&5&6&7&8\\4&8&5&1&3&7&6&2\end{pmatrix}, I.$ (j) $\begin{pmatrix}1&2&3&4&5&6&7&8\\7&3&2&6&8&4&1&5\end{pmatrix}, I.$

2. (a) $\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}, I.$ (b) $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}, I.$ (c) $\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, I.$

(d) $\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}, I.$ (e) $\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}, I.$

3. In 1(a), the first and third rotations listed in the answer are inverses, the second and fourth are their own inverses. In 1(c), the first two rotations listed are inverses. In 2(a), each rotation is its own inverse. In 2(c), the first two are inverses.

REVIEW EXERCISE 8 (page 238)

1. (a) Closure, associativity. (b) Closure. (c) All. (d) All.
(e) Closure, associativity. (f) All. (g) Closure, identity.

2. (a) All.
(b) Abelian group under addition; associativity of multiplication, distributivity.
(c) Associativity, identity and inverses under addition; closure, associativity under multiplication, distributivity.
(d) Abelian group under addition; associativity of multiplication, distributivity.
(e) All.
(f) Abelian group under addition; closure under multiplication.

3. (a) All. (b) 2(b), (c), (f).

4. 2(a) not a field; no multiplicative identity or inverses.
2(b) not a field; no closure under multiplication.
2(c) not a field; no closure under addition.
2(d) not a field; no closure under multiplication, no multiplicative identity or inverses.
2(e) not a field; no multiplicative identity or inverses.
2(f) not a field; no associativity of multiplication, no distributivity.

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